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Vol. 9, No. 9 October, 1948 pp. 483–558

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MATHEMATICAL REVIEWS
Published monthly, except August, by
THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

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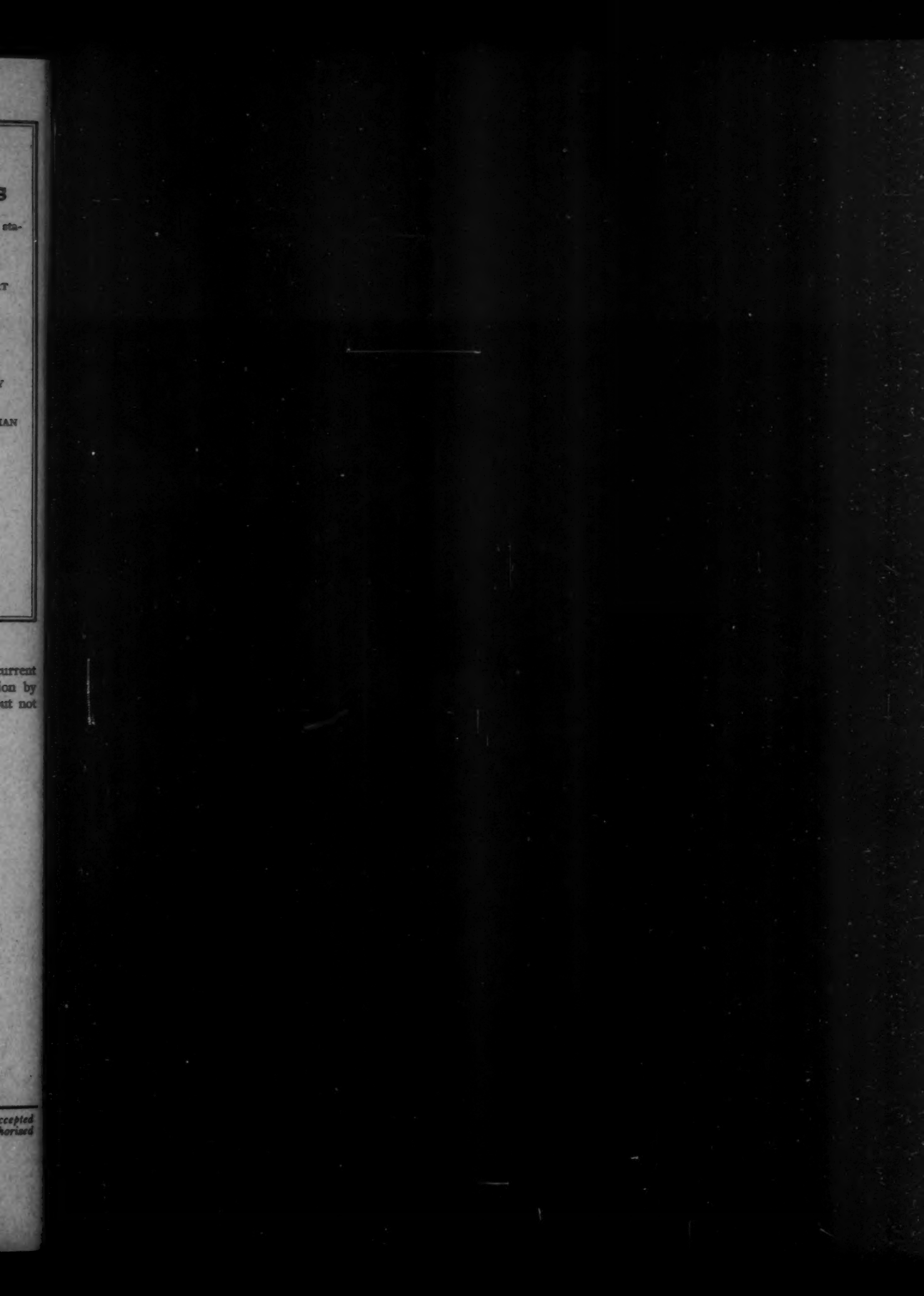
This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Naval Research, Department of the Navy, U.S.A. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.



Mathematical Reviews

Vol. 9, No. 9 OCTOBER 1948 Pages 483-558

HISTORY

*Neugebauer, O. **Arithmetical methods for the dating of Babylonian astronomical texts.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 265–275. Interscience Publishers, Inc., New York, 1948. $5.50.

The author previously showed [Quellen u. Studien z. Gesch. d. Math. Abt. B. 4, 34–91 (1937)] by using a Diophantine equation of the first degree that, if two astronomical cuneiform texts T_1 and T_2 contain the same astronomical phenomena represented by a linear zigzag function, it may be possible to date T_2 if the date of T_1 is known. The introduction to the present paper contains a brief outline of this method. The main problem to be solved in the present paper is the analogous one for a difference sequence of the second order formed by a summation over a linear zigzag function. A necessary condition for connectibility of two difference sequences of the second order is the connectibility of the corresponding linear zigzag functions. As for the sufficiency this question can be settled by actually carrying out the summation over the linear zigzag function. The author develops methods which facilitate the carrying out of such summations. [The reviewer found the following misprints: p. 268, formula (10) read: $y_k+y_{\tau-k}=2\mu$; p. 268, line 21, read: this relation remains correct even if the value μ does not occur; p. 269, formula (13) read: $\epsilon/d_\tau=[\epsilon/d_\tau]+\tau/d_\tau$; p. 269, formula (15a) replace c by $\pm c$, where the upper signs holds for $\tau\equiv 3 \pmod 4$, the lower for $\tau\equiv 1 \pmod 4$; p. 271, the first line of (17) should be $\cdots 2\Delta/d=[P]+d_1/d$; p. 271, lines 24–25, read: Hence the second interval will be normal if $d-d_1>d_1$.] *O. H. Schmidt* (Copenhagen).

Bortolotti, Ettore. Le fonti della matematica moderna. Matematica sumerica e matematica babilonese. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) 7, 77–97 (1940).

The author gives a summary of problems published by the reviewer in his edition of mathematical cuneiform texts [Quellen u. Studien z. Gesch. d. Math. Abt. A. 3 (1935)]. The historical conclusions, however, are new. From the fact that large series of examples for quadratic equations are made up from the same roots he concludes that these examples should verify "arcane proprietà mistiche" of this pair of roots. Babylonian mathematics is declared a degenerate form of Sumerian mathematics of which we have no texts. *O. Neugebauer* (Providence, R. I.).

Bortolotti, Ettore. Influenza del campo numerico sullo sviluppo delle teorie algebriche. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) 8, 3–11 (1941).

The author describes what he calls "Sumerian mathematics of the fourth millennium B.C.," then briefly discusses the Greek theory of geometrical algebra and ends with the discoveries of the school of Bologna, especially Bombelli. Many corrections to the first part could be made, e.g. fourth millennium should be replaced by second millennium. It is also wrong to deny the existence of approximations to irrational square roots, to assume a geometrical basis of the quadratic equations or to deny the existence of texts of this type in the Hellenistic period.

O. Neugebauer (Providence, R. I.).

Brtuins, E. M. Square roots in Babylonian and Greek mathematics. Nederl. Akad. Wetensch., Proc. 51, 332–341 = Indagationes Math. 10, 121–130 (1948).

The author discusses methods which might explain approximations of square roots preserved in ancient mathematical documents. For the approximation of $d=(a^2+b^2)^{\frac{1}{2}}$ he suggests finding λ such that $\lambda-\lambda^{-1}=2a/b$. Then $d=a+\lambda^{-1}b$. This method has the advantage of making use of the Babylonian tables of reciprocals. The author explains in this way the value $\sqrt{2}=1;25$ but he has obviously overlooked the better approximation 1;24,51,10 which was published by Neugebauer and Sachs [Mathematical Cuneiform Texts, New Haven, 1945; these Rev. 8, 1] and which also occurs in the Almagest [ed. Heiberg 35, 15]. The proposed method fails in this case. Geometric considerations are used to explain values found in Heron's and Archimedes's writings, leading to iterations closely related to the use of arithmetic and harmonic means. *O. Neugebauer.*

Please see Errata in Math. Rev. v.12, p. 1051

v. d. Waerden, B. L. Die Arithmetik der Pythagoreer. I. Math. Ann. 120, 127–153 (1948).

In diesem ersten Teil eines dreigliedrigen Aufsatzes wird gezeigt, wie man, auf einem von Tannery und Reidemeister zuerst gewiesenen Weg fortschreitend, durch Analyse der zahlentheoretischen Bücher Euklids (VII–IX) tiefer als bisher gelungen ist in die Arithmetik der Pythagoreer eindringen kann. Es ergibt sich, dass Buch VII, mit möglicher Ausnahme der letzten drei Sätze, von pythagoreischer Herkunft ist, ja geradezu die Grundlage der pythagoreischen Arithmetik bildet. Des weniger vollkommne Buch VIII rührt von Archytas her. Von Buch IX beruht die erste Hälfte auf Buch VIII, ist später entstanden und also nicht mehr pythagoreisch; dagegen ist der letzte Teil, die Lehre vom Geraden und Ungeraden, wie Becker und Reidemeister zu zeigen vermochten, altpythagoreisch. Vorgreifend teilt Verf. schon kurz die hauptsächlichsten Ergebnisse der zu erwartenden Teile II und III mit; in II soll die Beziehung der pythagoreischen Zahlentheorie zu der Lehre vom Irrationalen, in III die der pythagoreischen Mathematik zur babylonischen untersucht werden. *E. J. Dijksterhuis.*

*Michel, Henri. **Traité de l'Astrolabe.** Gauthier-Villars, Paris, 1947. viii+202 pp. (24 plates). 1200 francs.

The astrolabe is an instrument based on stereographic projection of the celestial sphere upon the plane of the equator. This instrument played a great role in the graphical solution of problems of spherical astronomy during the Middle Ages. In recent years the astrolabes have attracted much attention, as is shown, e.g., by the two monumental volumes "The Astrolabes of the World" by A. Gunther [Oxford, 1932]. The present work is less pretentious but

more reliable than Gunther's book. Its strength lies especially in the description of the Arabic astrolabes, their theory and history. Its main weakness lies in the discussion of the Greek material. The reader is misled [p. 47] to believe that nowhere in the ancient literature "on n'y trouve la démonstration des méthodes employées." Actually a complete discussion of the whole theory, both with graphical and trigonometrical methods, is given by Ptolemy in the "Planisphaerium." It is also wrong to say [p. 13] that the history of the astrolabe does not go further back than the 6th century A.D. This instrument is quoted 400 years earlier in a passage of the "Planisphaerium" [ed. Heiberg, p. 249, 22], and was used for mechanical clocks in the second century [Rehm, Jahresh. d. Österr. Archaeol. Inst. 6, 41–49 (1903)] as described by Vitruvius in the beginning of our era. The treatises of Philoponus and Severus Sebokht are based on a work of Theon Alexandrinus as shown by a detailed description of the latter by Ja'qūbī. For the treatise of Philoponus the very incompetent translation, published in Gunther's work, of a bad Greek text should be replaced by the excellent French translation of a much better text by Tannery [Mém. Sci. 9, 341–367]. Equally useful is Drecker's German translation and commentary [Isis 11, 15–44 (1928)]. *O. Neugebauer* (Providence, R. I.).

Luckey, P. Die Ausziehung der n-ten Wurzel und der binomische Lehrsatz in der islamischen Mathematik. Math. Ann. 120, 217–274 (1948).

A thorough account of the methods used by An-Nasawī [first half of 11th century] and Ghiyáth ad-Dín al-Káshí [first half of 15th century] for extracting the square, cube and higher roots of numbers. The author gives a conclusive proof of the fact that the Saracens knew two methods of finding the nth root, viz., (1) the method used by the Indians, Europeans at the time of the Renaissance and, presumably, 'Umar-i Khaiyámí, based on the binomial formula $(a+b)^n$; (2) Horner's method by successive divisions, used by both Nasawí and Káshí and even Gúshyár [about 1000 A.D.]. The paper further contains a discussion of the binomial theorem as it occurs in Káshí's Arithmetic.

A. C. M. Overing (Flushing, Holland).

***Sarton, George. Introduction to the History of Science. Volume III. Science and Learning in the Fourteenth Century.** Carnegie Institution of Washington, Publication 376. Part 1, pp. i–xxxv+1–1018; part 2, pp. i–xiv+1019–2155. 1947.

This is the concluding part of a work which has long become an indispensable bibliographical tool of historians of science. [The first volume appeared in 1927, the second in 1931.] *O. Neugebauer* (Providence, R. I.).

***Gnedenko, B. V. Očerki po Istorii Matematiki v Rossii. [Outline of the History of Mathematics in Russia].** OGIZ, Moscow-Leningrad, 1946. 247 pp.

Written for a "wide circle of readers" in the USSR, this book will be of interest to all who seek to explain the tremendous volume of current Soviet mathematical research. The author does not confine himself to a recital of names, dates, and topics, but attempts successfully to characterize the work of individuals and schools by citing typical theorems and applications, where such background material is susceptible of assimilation by the nonspecialist reader. The book's three chapters discuss: (1) mathematics in Russia before the 18th century [58 pages], (2) the 18th and 19th centuries [89 pages], (3) the 20th century [70 pages].

Chapter 1 considers the cultural influence of the medieval church, the role of calendariography and a system of numeration based on the Church-Slavonic alphabet. In this chapter the author restricts himself to the Ukraine and Russia proper, leaving aside the important medieval contributions of Central Asia. Facsimile pages and typical problems from several 17th century manuscripts are given. The foundation of new schools by Peter the Great is noted, the chapter closing with a discussion of the influential "Arifmetica" [printed in 1703] of Magnickiĭ, a sort of compendium of elementary mathematics.

Chapter 2 sketches the founding and vicissitudes of the St. Petersburg Academy of Sciences, a section being devoted to the life and work of Euler. After noting the founding of additional universities at Moscow, Kazan, Har'kov (Kharkov), and Kiev, another section is given over to Lobačevskiĭ. In a description of the development of non-Euclidean geometry which is included, the name of Bolyai does not appear, and Gauss is mentioned only as being the sole mathematician contemporary with Lobačevskiĭ who appreciated his work. There follows a characterization of the "St. Petersburg school" of mathematicians, including mention of the work of V. Ya. Bunyakovskiĭ, followed by a more extensive outline of that of M. V. Ostrogradskiĭ (1801–1861). Many of the latter's results, unknown in western Europe, were subsequently rediscovered by others. The next section is devoted to P. L. Čebyšev (Tschebicheff, etc.) (1821–1894), whose interests are outlined. Of two of Čebyšev's pupils next noted, the first is A. A. Markov (1856–1922). The second is A. M. Lyapunov (1857–1918). Prominence is given to his study of the stability of the piriform body, in which he anticipated Poincaré. The chapter closes with an account of the life of Sof'ya Kovalevskaya (Sonja Kowalewski, etc.) (1850–1891), and a sketch of the beginnings of the Moscow Mathematical Society.

Chapter 3 describes the impetus given to the study of mathematics by the change of political régime. The number of universities has more than doubled since the revolution, research institutes have been started and teaching methods overhauled, great emphasis being placed on seminars. The closing sections of the book are devoted to the Soviet schools of number theory and probability theory, respectively.

Four appendices of an expository nature deal with: Slavonic numeration, the curve enclosing maximum area, Čebyšev polynomials and measure. There is no index, and the bibliography lists only works in Russian.

E. S. Kennedy (Beirut).

Käfer, Karl. Der Kettensatz. Ein Beitrag zur Geschichte und Theorie des kaufmännischen Rechnens. Mitt. Handelswiss. Sem. Univ. Zürich. N.F. 68, 415 pp. (1941).

Die Arbeit bildet einen Beitrag zur Entwicklungsgeschichte des praktischen Rechnens. Zuerst wird auf Grund der Quellen ein zusammenhängendes Bild der Entwicklung des unter dem Namen Kettensatz bekannten kaufmännischen Rechnenverfahrens von seinen Anfängen bis zum heutigen Stand gegeben. Dabei werden auch verwandte Rechenverfahren, vor allem der Dreisatz und der Vielsatz berücksichtigt. Sodann werden in zweiten Teil Name, Anwendungsgebiet, Darstellungsform und Ausrechnungsverfahren des Kettensatzes in ihrer Entwicklung systematisch behandelt, besonders im Hinblick auf die Ausbildung der heutigen Methoden. Im dritten Teil werden die bisherigen

Begründungen auf ihre Stichhaltigkeit geprüft; es wird untersucht, unter welchen Voraussetzungen und für welche Aufgaben der Kettensatz angewendet werden kann und wie er zu beweisen ist. Es ergibt sich das Resultat, dass das Anwendungsgebiet des Satzes wesentlich grösser ist als üblicherweise zugestanden wird und dass ein Beweis mit vollständiger Klarstellung aller Voraussetzungen und Anwendungsmöglichkeiten mit Hilfe von einfachen Sätzen aus der Gleichungs- und Funktionenlehre gegeben werden kann.
E. J. Dijksterhuis (Oisterwijk).

Boyer, Carl B. Cartesian geometry from Fermat to Lacroix. Scripta Math. **13**, 133–153 (1947).

Coolidge, J. L. The beginnings of analytic geometry in three dimensions. Amer. Math. Monthly **55**, 76–86 (1948).

Godeaux, Lucien. La géométrie algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 901–918 (1947).
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Richeson, A. W. The condition of English mathematics from 1750–1850. Math. Student **14** (1946), 49–57 (1948).

Gloden, A. Aperçu historique sur les séries. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. **17**, 113–120 (1947).

Golab, Stanislaw. Il contributo dei matematici polacchi contemporanei alla scienza mondiale. Boll. Un. Mat. Ital. (3) **2**, 244–251 (1947).

***Archibald, Raymond Clare. Mathematical Table Makers. Portraits, Paintings, Busts, Monuments, Bio-Bibliographical Notes.** Scripta Mathematica Studies, no. 3. Scripta Mathematica, New York, N. Y., 1948. iv+82 pp. (20 plates). $2.00.
An enlarged version of two articles which appeared in Scripta Math. **11**, 213–245 (1945); **12**, 15–51 (1946); these Rev. **8**, 3. Fifty-three table makers are considered.

***Smart, W. M. John Couch Adams and the Discovery of Neptune.** Royal Astronomical Society, London, 1947. i+56 pp. (3 plates). 5s.

Puppini, Umberto. La forma originaria del teorema di Daniel Bernoulli nell'idrodinamica. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **10**, 75–86 (1943).

Bachmann, E. Zum 200. Todestag des Mathematikers Johann Bernoulli. Schweiz. Z. Vermessg. Kulturtech. **46**, 125–128 (1948).

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***Copernicus, Nicolaus. De Revolutionibus. Preface and Book I.** Translated by John F. Dobson assisted by Selig Brodetsky with a biographical note and notes to the translation. Royal Astronomical Society, London, 1947. 32 pp. (2 plates). 3/6.

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Reproduction and discussion of Desargues' original statement and proofs.

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Severi, Francesco. Obituary: Michele de Franchis. Revista Mat. Hisp.-Amer. (4) **7**, 279–281 (1947). (Spanish)

Obituary: Guido Fubini Ghiron. Ann. Mat. Pura Appl. (4) **25**, ix–xii (1946).

***Lambert, Johann Heinrich. Iohannis Henrici Lamberti Opera Mathematica. Volumen Secundum. Commentationes Arithmeticae, Algebraicae et Analyticae, Pars Altera. Edidit Andreas Speiser.** Orell Füssli, Zürich, 1948. xxix+324 pp.
This is the concluding volume of Lambert's collected works on pure mathematics [for vol. 1, published in 1946, cf. these Rev. **8**, 306]. A volume on applied mathematics is planned. Even the present volume contains much on practical computation, including tables. The editor again contributes explanatory remarks to each paper, among which no. 2 on the irrationality of π and no. 9 on the Lambert series might be especially mentioned. *O. Neugebauer.*

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For part I cf. the same vol., 1–15 (1947); these Rev. 9, 169.

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Hommage à la mémoire de l'éminent assyriologue François Thureau-Dangin (1872–1944). Mededeelingen en Verhandelingen van het Vooraziatisch-Egyptisch Gezelschap "Ex Oriente Lux," no. 8, 35 pp. (1 plate) (1946).

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FOUNDATIONS

McKinsey, J. C. C., and Tarski, Alfred. Some theorems about the sentential calculi of Lewis and Heyting. J. Symbolic Logic 13, 1–15 (1948).
The results of two papers by the same authors [Ann. of Math. (2) 45, 141–191 (1944); 47, 122–162 (1946); these Rev. 5, 211; 7, 359] are applied to logical calculi. With every formula α of the Lewis system S4, containing at most the variables $v_1, \cdots, v_n$, there is associated a closure-algebraic function $f^{(\alpha)}(x_1, \cdots, x_n)$; the definition of $f^{(\alpha)}$ follows the well-known process of evaluation of α in a matrix, the operations $\cdot$, $-$, C of closure-algebra corresponding to conjunction, negation and possibility in S4. Then [theorem 1.4] the following conditions are equivalent: (i) α is provable in S4; (ii) $f^{(\alpha)}$ is identically equal to 1 in every closure-algebra. Moreover, if A is any dissectable closure-algebra, then (i) and (ii) are equivalent to (iii) $f^{(\alpha)}$ is identically equal to 1 in A. Thus A can be considered as a characteristic matrix for S4. From this theorem follows immediately: (a) if α is provable in S4, $\sim\Diamond\sim\alpha$ is provable in S4; (b) if $\sim\Diamond\sim\alpha \vee \sim\Diamond\sim\beta$ is provable in S4, then either α or β is provable in S4.

An extension of S4 is called normal if it satisfies (a); S5 is a normal extension of S4. A necessary and sufficient condition that α be provable in S5 is that $f^{(\alpha)}$ be identically 1 in every closure-algebra in which every closed element is also open. Theorems are proved from which it follows that neither S4 nor S5 has a finite characteristic matrix.

An analogue of theorem 1.4 is obtained if we substitute the intuitionistic calculus of propositions H for S4 and Brouwerian algebra (Brouwerian-algebraic function) for closure-algebra (closure-algebraic function). Here $\cdot$, $+$, $\dotplus$ and $\neg$ of Brouwerian algebra correspond to $\wedge$, $\vee$, $\rightarrow$ and negation in H. Corollary: if $\alpha \vee \beta$ is provable in H, then either α or β is provable in H. [Reviewer's note: this was proved by Gentzen, Math. Z. 39, 405–431 (1934); in particluar, p. 407.] There exist in H infinitely many non-equivalent formulas involving only one variable and only the signs "$\rightarrow$", "$\vee$" and negation. An improvement is given of the known result that no finite characteristic matrix for H exists. Finally, there is correlated in three different ways with every formula α of H a formula $T(\alpha)$ of S4, so that α is provable in H if and only if $T(\alpha)$ is provable in S4.
A. Heyting (Amsterdam).

Popper, K. R. Logic without assumptions. Proc. Aristotelian Soc. N.S. 47, 251–292 (1947).
This paper discusses the philosophical background of the author's inferential formulation of the logic of propositions and predicates. [See Mind 56, 193–235 (1947); 57, 69–70 (1948); Nederl. Akad. Wetensch., Proc. 50, 1214–1224 = Indagationes Math. 9, 561–571 (1947); these Rev. 9, 130, 321; cf. also the papers reviewed below.] It contains no mathematical results; but it throws light on the motivation of the author's other papers. The major point is that logic deals with formative signs, and the latter should be definable in terms of pure deducibility. *H. B. Curry.*

Popper, K. R. On the theory of deduction. I. Derivation and its generalizations. Nederl. Akad. Wetensch., Proc. 51, 173–183 = Indagationes Math. 10, 44–54 (1948).
In this paper the author revises the system of inferential propositional logic in his previous papers [cited in the preceding review]. The principal innovations are as follows. (1) He derives the principle Tg from Gentzen's elimination

rule (Schnitt) without mentioning that Gentzen did this [Math. Ann. 107, 329–350 (1932), in particular, p. 332]. Aside from certain idiosyncrasies of the author's formulation, the resulting basis, called basis I, differs from Gentzen's [loc. cit.; or that formed by only the "Struktur-Schluss Figuren" of Math. Z. 39, 176–210 (1934), in particular, p. 192] chiefly in that Gentzen's rule of commutation [Vertauschung in loc. cit. 1934; tacit in loc. cit. 1932] is replaced by more specific assumptions. (2) He remarks that the change in (1) removes the need for basis II, and that his "basic definitions" can be formulated on basis I. Nevertheless he claims basis II has some interest in its own right and makes some minor improvements in it; but he does not correct the error mentioned in the review of the second of his papers quoted above.

(3) He introduces Gentzen's [loc. cit. 1934] sequences with multiple consequents (along with some related notions), essentially by the definition

$$a_1, \cdots, a_m \vdash b_1, \cdots, b_n \rightleftarrows (c)[b_1 | c \,\&\cdots\&\, b_n | c \rightarrow a_1, \cdots, a_m | c].$$

Here "(c)" indicates universal quantification, "$\rightarrow$" implication, and "&" conjunction on the metalinguistic level: such notions are characteristic of the author's work. The principal difference between the resulting system and that of Gentzen [loc. cit. 1934] is that the "Logische-Zeichen-Schluss Figuren" are replaced by quasi-definitions of which the following variants of essentially the same example are typical:

$$a | b \supset c \rightleftarrows a, b | c,$$
$$a \| b \supset c \rightleftarrows (d)[d | a \rightleftarrows d, b | c].$$

Here "$\|$" indicates a kind of equivalence. The system lacks the noneliminative character which leads to Gentzen's Hauptsatz. There may well be other, less obvious differences also. *H. B. Curry* (State College, Pa.).

Popper, K. R. On the theory of deduction. II. The definitions of classical and intuitionist negation. Nederl. Akad. Wetensch., Proc. 51, 322–331 = Indagationes Math. 10, 111–120 (1948).

This is an analysis of negation on the basis of the author's paper reviewed above. The author considers 6 kinds of negation, viz., a^i, a^m, a^k, a^j, a^l, a^u, with properties as follows:

$$a \vdash b^i \rightleftarrows a, b \vdash; \quad a^m \vdash b \rightleftarrows \vdash a, b; \quad a, a_k \vdash \,\& \vdash a, a_k;$$
$$a, b \vdash c^j \rightarrow a, c \vdash b^j; \quad a, b^l \vdash c \rightarrow a, c^l \vdash b; \quad a, b \vdash c \rightarrow a, c^u \vdash b^u;$$

of these a^i is intuitionist negation, a^m is a form of complementation dual to a^i, a^k is classical negation, a^j the minimal negation of Johansson [Compositio Math. 4, 119–136 (1936)], a^l its dual, and a^u a still weaker form. Of the six, the postulates for a^i, a^m, a^k determine them uniquely if they exist (so that these postulates are quasi-definitions), while for the last three this is not the case. The author argues from this, on the basis of the philosophy in the paper of the second preceding review, that these are not "formative signs" and therefore not strictly logical. Also if a^k exists in any language, then $a^i = a^m = a^k$; but the author uses three-valued logic as an example to show that a^i and a^m may both exist without there being an a^k. Various alternative quasi-definitions for a^k are considered; also connections with a modal notion of impossibility. At the close there are remarks concerning the existential character of assumptions in regard to negation. *H. B. Curry* (State College, Pa.).

Kemeny, John G. Models of logical systems. J. Symbolic Logic 13, 16–30 (1948).

Verf. definiert, in der Sprache der naiven (d.h. nicht formalisierten axiomatischen) Mengenlehre, einen allgemeinen Begriff des "logischen Systems" und des "Modells für ein logisches System." Trotz der grundsätzlichen Widersinnigkeit metamathematischer Untersuchungen in einer mathematischen Sprache wird dadurch eine nützliche, vorläufige Klärung erreicht.

Ein logisches System wird, wie bei Tarski [Monatsh. Math. Phys. 37, 361–404 (1930)] eingeführt als eine Menge von Zeichen, für die gewisse Zeichenreihen als "Ausdrücke" und ausserdem endlich viele Regeln der Form "$\mathfrak{a}_1, \cdots, \mathfrak{a}_n$ folglich $\mathfrak{a}$" gegeben sind. Die nach diesen Regeln ableitbaren Ausdrücke heissen "Sätze." Es wird ferner gefordert, dass die Zeichenmenge aus λ, (,), Variablen und Konstanten bestimmter Typen besteht, so dass jeder Ausdruck eine "Formel vom Typ σ" ist, wenn definiert wird: (1) o, ι, ι', ι'', $\cdots$ sind Typen; (2) sind α, β Typen, so ist $(\alpha\beta)$ ein Typ; (3) Variable und Konstante sind Formeln ihres Typs; (4) ist $\mathfrak{x}$ eine Variable von Typ β, $\mathfrak{a}$ eine Formel vom Typ α, so ist $(\lambda\mathfrak{x}\mathfrak{a})$ eine Formel vom Typ $(\alpha\beta)$; (5) ist $\mathfrak{a}$ eine Formel vom Typ α, $\mathfrak{b}$ eine Formel vom Typ β, so ist $(\mathfrak{a}\mathfrak{b})$ eine Formel vom Typ α.

Ein (zweiwertiges) Modell eines logischen Systems L wird definiert als eine Menge M, die enthält: (1) die Menge der "Wahrheitwerte" $R_o = \{t, f\}$; (2) beliebige "Individuenbereiche" $R_\iota, R_{\iota'}, R_{\iota''}, \cdots$; (3) für alle Typen $(\alpha\beta)$ die Menge $R_{(\alpha\beta)}$ aller Funktionen über R_β mit Werten in R_α; und zu der eine umkehrbare Zuordnung gegeben ist, die jeder Konstanten vom Typ α von L ein Element von R_α zuordnet. Präzisiert man die übliche "Deutung" von $(\lambda\mathfrak{x}\mathfrak{a})$ als "die durch $\mathfrak{a}$ definierte Funktion von $\mathfrak{x}$" und von $(\mathfrak{a}\mathfrak{b})$ als "den Funktionwert von $\mathfrak{a}$ an der Stelle $\mathfrak{b}$," so wird durch jeden Ausdruck von L eine Funktion von M mit Werten in R_o "ausgedrückt."

Ein Ausdruck von L heisst "in M gültig," wenn "die ausgedrückte Wahrheitsfunktion" gleich der Konstanten t ist. Von M wird gefordert, dass für jede Regel $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \rightarrow \mathfrak{a}$ von L mit $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$ stets auch $\mathfrak{a}$ in M gültig ist.

Mit Hilfe dieses Modellbegriffs wird dann definiert: (1) Ein logisches System L heisst widerspruchsfrei, wenn es ein Modell M von L gibt, für das nicht jeder Ausdruck von L in M gilt. (2) Ein logisches System heisst vollständig, wenn es ein Modell M von L gibt, so dass ein Ausdruck von L genau dann ein Satz von L ist, wenn er in M gilt. (3) Ein System L' heisst gleichwertig mit L, wenn jedes Modell M' von L' in einem Modell von L enthalten ist, so dass jedes in L' ausdrückbare Element von M' auch in L ausdrückbar ist, und umgekehrt. Verf. gibt für diese Begriffe Anwendungsbeispiele an den formalistischen und intuitionistischen Logikkalkülen. *P. Lorenzen* (Bonn).

***Chwistek, Léon. La méthode générale des sciences positives. L'esprit de la sémantique.** Actualités Sci. Ind., no. 1014. Hermann et Cie., Paris, 1946. 43 pp.

This is a condensed statement of the author's views on the foundations of logic and mathematics, based on a slightly revised version of his formal system of metamathematics. This system, called "rational semantics," is expounded briefly in the third and fourth chapters of the present booklet. It has previously been described elsewhere in greater detail [see L. Chwistek and W. Hetper, J. Symbolic Logic 3, 1–36 (1938)]. The essential features of the formal system are unchanged. There are two primitive symbols,

the letter c and the dyadic operator $*$, which is applied in the manner of Łukasiewicz to avoid the need for parentheses. Other symbols are introduced as abbreviations for certain combinations of these two. One such combination replaces the Sheffer stroke function, and another, abbreviated as $(EFGH)$, has the interpretation "H is the result of substituting G for F in E." This substitution symbol is the characteristic feature of Chwistek's system. There are also symbols for the universal quantifier in each type, and for the positive integers in each type. The ramified theory of types is retained. This stratification provides, for each subsystem of fixed type, a metalanguage of higher type, and allows for a Gödel enumeration of formulas and for Gödel's theorem of indeterminacy. A theory of classes and of relations is sketched.

In the first two chapters and in the introduction, the author defines his philosophical position. He calls himself a positivist, makes conformity with common sense his first criterion, and opposes what he calls the idealism of Cantor. He criticizes the formalists because they retain Cantorism, but his system is formalistic in the extreme. It is called rational semantics, but the system as a whole is purely syntactical, the only semantics involved being that implied by the syntax or provided by the intended interpretation of the symbols. The author attempts to answer Church's criticism [J. Symbolic Logic **2**, 168–170 (1937)] that his failure to distinguish between the use and mention of expressions involves an ambiguity of interpretation.

The last two chapters of the booklet deal with the implications of the author's theory in geometry and in the natural sciences. *O. Frink* (State College, Pa.).

Billing, J. A failure of the Bolzano-Weierstrass lemma. Ark. Mat. Astr. Fys. **34B**, no. 11, 2 pp. (1947).

The author points out that the Bolzano-Weierstrass theorem fails in the case of a certain sequence, unless one accepts what amounts to the classical law of excluded middle. The argument appears to be essentially intuitionistic.
R. M. Martin (Philadelphia, Pa.).

Borel, Émile. À propos de l'axiome du choix. Ark. Mat. Astr. Fys. **34B**, no. 15, 2 pp. (1948).

Borel argues that the failure of the Bolzano-Weierstrass theorem noted by Billing in the paper reviewed above rests upon the paradoxical nature of the axiom of choice.
R. M. Martin (Philadelphia, Pa.).

***Polya, G. How to Solve It. A New Aspect of Mathematical Method.** Princeton University Press, Princeton, N. J., 1948. xv+224 pp. $3.00.

[This is an extended version of a book which was first published in 1945; cf. these Rev. **6**, 198.] Searching for answers to problems that arise in the conduct of our lives or arouse our curiosity is a constant occupation of our minds. Philosophers and mathematicians have tried to devise rules for the direction of mind in this its effort, as witnessed by Pappus's report on the ancient art of heuristics, by Descartes's and Leibniz's attempts, and those of many minor lights. Indeed, if there are rules for truth (logic), one may expect that there are also rules for finding the truth. How much thought the author has given to the ways and means by which the problem-solving capacity of a mathematician in statu nascendi may be trained is evidenced by the preface and the two volumes of Pólya and Szegö [Aufgaben und Lehrsätze aus der Analysis, Springer, Berlin, 1925]. The present book, a sort of elementary textbook on heuristic reasoning, shows anew how keen its author is on questions of method and the formulation of methodological principles. His standpoint here is chiefly pedagogical; heuristics is conceived as a half-logical, half-psychological affair. Exposition and illustrative material are of a disarmingly elementary character, but very carefully thought out and selected.

A one-page, concise list of questions and suggestions about the four phases of Understanding the problem, Devising a plan, Carrying out the plan, Looking back, opens and ends the book. Its main part is in the form of a dictionary. Analysis, auxiliary elements, generalization, induction and mathematical induction, are among the items discussed, but there are many less obvious ones. Here are a few of the author's characteristic aphorisms. "It is foolish to answer a question that you do not understand. It is sad to work for an end that you do not desire." "Good ideas are based on past experience and formerly acquired knowledge." "Rules of style: The first rule of style is to have something to say. The second rule of style is to control yourself when, by chance, you have two things to say; say first one, then the other, not both at the same time." "The first rule of teaching is to know what you are supposed to teach. The second rule of teaching is to know a little more than you are supposed to teach." *H. Weyl* (Princeton, N. J.).

***Bouligand, Georges, et Desbats, Jean. La Mathématique et son Unité. Introduction aux Éléments de l'Analyse et à la Philosophie des Sciences Déductives.** Payot, Paris, 1947. 311 pp. 630 francs.

The authors present selected topics in the history of mathematics, in elementary college mathematics and in logic and use these topics to show how mathematics unifies processes of reasoning. *C. C. Torrance* (Annapolis, Md.).

Benda, Julien. L'intuition et la mathématique. À propos d'un livre récent. Rev. Gén. Sci. Pures Appl. N.S. **54**, 193–198 (1947).

Comment on the book of Bouligand and Desbats reviewed above.

Vietoris, L. Über den Begriff der Wahrscheinlichkeit. Monatsh. Math. **52**, 55–85 (1948).

"There is at present a very general conviction that one cannot define the concept of probability entirely in terms of mathematical and logical concepts, but that a further concept is required, and that this must be introduced as an undefinable fundamental concept and described by means of axioms." With this opening sentence, the author explains his reasons for choosing for this fundamental concept the ordering relation $\leqslant$ ($a_A \leqslant b_B$ corresponding with the intuitive idea "outcome a in trial A is not more probable than outcome b in trial B"), rather than the (numerical) probability of an event, a less intuitively natural notion, which should appear as derived from the former. The object of the paper is then carried out: the introduction of nine axioms governing $\leqslant$ and the derivation of the laws of classical probability from these.

Axiom (I) makes the system of symbols a_A a completely ordered one. Axiom (II) states that if $a_A \leqslant b_B$ then $\bar{b}_B \leqslant \bar{a}_A$ ($\bar{a}$ denoting the contrary of a, etc.). Axiom (III) says that, if a_A logically implies b_B, then $a_A \leqslant b_B$. Let a and a' be mutually exclusive outcomes in A, and b and b' exclusive outcomes in B. Then axiom (IV) states that, if $a_A \leqslant b_B$ and

$a_A' \leqslant b_B'$, then $(a \vee a')_A \leqslant (b \vee b')_B$; and axiom (V), that if $a_A \leqslant b_B$ and $a_A' < b_B'$, then $(a \vee a')_A < (b \vee b')_B$. Here $\vee$ is the logical sum and $<$ is the presence of $\leqslant$ and absence of $\geqslant$. Axiom (VI): an outcome certain to occur has the relation $>$ to an impossible outcome. Axiom (VII): an outcome certain in trial A has the relations $\geqslant$ and $\leqslant$ to such a one in trial B.

What we have here called "trials" the author calls "games," a notion which he amplifies in all formal precision. He introduces, by illustrations and by a general description, his idea of the group of a game (when the game has enough symmetry). Then axiom (VIII) states that, if a and b are congruent under the group of A, $a_A \geqslant b_A$ and $a_A \leqslant b_A$. This allows a Laplacian scheme to be given for the numerical definition of probabilities in games having transitive groups and, by comparison and passage to limits, in any other game. The system of axioms is completed with (IX): Let c and d be outcomes in A^k (the k-fold repetition of games like A) and let $c \leqslant d$; let a be an outcome in A. Then on a $(k+1)$-fold repetitive game (a with c) $\leqslant$ (a with d).

As the author points out clearly in his first footnote, his approach to probability coincides with the reviewer's [B. O. Koopman, Ann. of Math. (2) **41**, 269–292 (1940); **42**, 169–187 (1941); these Rev. **1**, 245; **2**, 227]. The differences are (1) in scope: the author's is less general than the reviewer's, chiefly because the latter considers partially ordered systems with regard to $\leqslant$ as the general case; (2) in method: the author uses the group of a game to get numerical probabilities; the reviewer used probability scales, the existence of which he axiomatized.

We find difficulty with the notion of the group of a game, not as abstract concept, but in any concrete application. If absolute symmetry is required, the game (if one existed!) would get into trouble with determinism: if all the conditions in the universe were symmetric with regard to the tossing of a coin, it would come down on an edge and spin on it through all eternity. If the game were symmetric only by ignoring certain factors (as is always done), there might be danger of ignoring the wrong ones unless the conditions under which they could be ignored were given; and such a statement of conditions would seem to have to be made in language involving the $\leqslant$ relation. But then axiom (VIII) would seem to be a theorem, deducible from the other axioms. *B. O. Koopman* (New York, N. Y.).

ALGEBRA

Kulebakin, V. S. On the application of the principle of absolute invariance to real physical systems. Doklady Akad. Nauk SSSR (N.S.) **60**, 231–234 (1948). (Russian)

N. N. Lusin and P. I. Kouznetzoff have determined necessary and sufficient conditions in order that in the linear system with constant coefficients

$$\text{(S)} \qquad \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = f(t), \\ a_{i1}x_1 + \cdots \qquad\quad = 0, \end{cases} \qquad i = 2, \cdots, n,$$

the solution x_1 be independent of $f(t)$. The author discusses a certain number of circuits à la Wheatstone bridge which give rise to systems of this nature. Thus in the bridge one of the currents x_1 is independent of the outside e.m.f. The problem is extended to the case where in (S) the right-hand sides are $f_i(t)$, $i = 1, 2, \cdots, n$, with x_1 to be independent of $f_1(t)$. An application is given to the regulation of the velocity of an electric d.c. motor with independent excitation. [See Lusin, Avtomatika i Telemehanika **1940**, 4–66 (1940); Lusin and Kouznetzoff, C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 251–253, 335–337 (1946); these Rev. **8**, 273, 73]. *S. Lefschetz* (Princeton, N. J.).

Kreis, H. Über die Grundfunktionen positiver Zahlen. Elemente der Math. **3**, 59–61 (1948).

Let $m_r(a_1, \cdots, a_n)$ denote the positive rth root of the quotient $E_r/\binom{n}{r}$, where E_r is the elementary symmetric polynomial of degree r in the n positive indeterminates $a_1, \cdots, a_n$ and $\binom{n}{r}$ is the number of combinations of n things r at a time. The finite sequence m_r $(r = 1, 2, \cdots, n)$ is known to be nonincreasing [Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, p. 51]. The present paper gives a neat elementary proof of this result by induction based on equalities $m_r(a_1, \cdots, a_n) = m_r(b_1, \cdots, b_{n+1})$ $(r = 1, 2, \cdots, n)$, where the b's are arbitrary positive numbers and the corresponding a's are the positive numbers got by changing the signs of the roots of $\sum_{i=1}^{n+1} 1/(x+b_i)$.

J. M. Thomas (Durham, N. C.).

Tietze, Heinrich. Über symmetrische Funktionen von abzählbar unendlich vielen Argumenten. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 43–48 (1944).

This deals with the fundamental theorem of symmetric functions of n complex arguments $x_1, x_2, \cdots, x_n$ when $n \to \infty$. Previously [Monatsh. Math. Phys. **48**, 487–499 (1939); **49**, 1–52 (1940); these Rev. **1**, 97, 323] the author proved the theorem, but only if $\sum_\nu |x_\nu|$ converges. Namely, any monomial symmetric function $\sum x_1^{b_1} \cdots x_n^{b_n}$ (all integers $b_i \geqq 0$, $\sum b_i = n$) is expressible, and expressible in only one way, as a polynomial, with integral coefficients, in the elementary symmetric functions $\sum x_1$, $\sum x_1x_2$, etc. When this convergence breaks down it may happen that $\sum |x_\nu|^m$ converges for some least positive integer $m > 1$. In this case the elementary symmetric functions also break down, but the sums of powers $s_m, s_{m+1}, \cdots$ avail, where $s_m = \sum x_\nu^m$. Two theorems state (1) that the above monomial function is absolutely convergent if and only if each exponent b_i either vanishes or is not less than m, and (2) that each such monomial function is expressible as a polynomial, with rational coefficients, in $s_m, s_{m+1}, \cdots, s_n$, where $n = \sum b_i \geqq m$. This means that, of the three basic types of symmetric function, the elementary, the Wronskian, and the sums of powers, the latter are the more fundamental when $n \to \infty$.

H. W. Turnbull (St. Andrews).

Motzkin, Th. Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products. Bull. Amer. Math. Soc. **54**, 352–360 (1948).

The Catalan numbers $C_{2n,n}/(n+1)$, introduced originally to enumerate the number of different products of $n+1$ factors, in given order, in a nonassociative multiplication, are given a new interpretation needed by the author in his study of the hypersurface cross ratio: the number of divisions of $2n$ points on a circle into n pairs without crossing. These are generalized to the number of divisions without crossings of nl points on a circle into n l-tuples and its elaboration and to certain ballot type problems (M always leads N in

a voting return). The generalization of a related problem of partitioning a convex polygon by nonintersecting diagonals which is also given has been anticipated by I. M. H. Etherington [Edinburgh Math. Notes no. 32, 1–6 (1941); these Rev. 4, 68]. *J. Riordan* (New York, N. Y.).

Motzkin, Th. Independence of resultants. Bull. Amer. Math. Soc. 54, 360–365 (1948).

A system of $m>n$ algebraic forms, $a_1, \cdots, a_m$ in n variables, taken in combinations of n forms, has $\binom{m}{r}$ resultants. The note proves that these resultants are independent functions of the coefficients in the forms provided that the degree d_k of a_k exceeds a bound depending only on n and k. The bound for which the theorem is proved is $d_k \geqq \binom{k-1}{n-1}$. *D. E. Littlewood* (Bangor).

Bruins, E. M. On the comitants of binary quadratic and cubic forms. Nederl. Akad. Wetensch., Proc. 51, 184–190 = Indagationes Math. 10, 55–61 (1948).

By proving and using certain theorems concerning the reducibility of concomitants of concomitants the author obtains the irreducible systems for certain combinations of binary quadratic and cubic forms. For one cubic and two quadratics he obtains more easily the irreducible system previously obtained by Spierenburg [Thesis, Amsterdam, 1936] and consisting of 4 cubic, 6 quadratic, 13 linear forms and 18 invariants. For two cubics and one quadratic he obtains the irreducible system: 1 quartic, 8 cubic, 12 quadratic, 20 linear forms, 27 invariants. *D. E. Littlewood.*

Abstract Algebra

Riguet, Jacques. Sur les lattices pseudo-modulaires. C. R. Acad. Sci. Paris 226, 1151–1153 (1948).

A lattice is called "left pseudo-modular" if $a \cap b = 0$ and $a \leqq c$ imply $(a \cup b) \cap c = a \cup (b \cap c)$. Some properties of these and related lattices and their decompositions are studied. Theorem: if a lattice L has a largest element I, and has an extra operation "multiplication" which is distributive with respect to $\cup$ and which has I as unit element, then $a \cup b = I$ implies $(a \cap b) \cup c = (a \cup c) \cap (b \cup c)$, and L is right pseudo-modular. *P. M. Whitman* (Silver Spring, Md.).

Sikorski, Roman. Sur les corps de Boole topologiques. C. R. Acad. Sci. Paris 226, 1675–1676 (1948).

A "topological Boolean algebra" is a closure algebra in the sense of McKinsey and Tarski [Ann. of Math. (2) 45, 141–191 (1944); these Rev. 5, 211], except that the author assumes countable additivity; however, no reference is made to McKinsey and Tarski. The author announces three theorems which give representations of such abstract algebras as algebras of sets. *I. Kaplansky.*

Sikorski, Roman. Sur la convergence des suites d'homomorphies. C. R. Acad. Sci. Paris 226, 1792–1793 (1948).

This is a continuation of the note reviewed above. The author proposes a topology for the set of σ-homomorphisms of a σ-Boolean algebra into a "topological" Boolean algebra and announces three theorems concerning this topology. *I. Kaplansky* (Princeton, N. J.).

Habicht, Walter. Zur inhomogenen Eliminationstheorie. Comment. Math. Helv. 21, 79–98 (1948).

The author studies elements contained in the polynomial ideal $(f_1, \cdots, f_n)$ which are linear in some of the variables. The coefficients are indeterminates. If the f_i depend on $n-1$ variables the ideal contains linear polynomials which are not multiples of the resultant. If the f_i depend on n variables ($n \geqq 2$), the author demonstrates the existence of n polynomials: $F_i = c_i x_i + d_i(x_n)$, $i = 1, \cdots, n-1$; $F_n = d_n(x_n)$, which (apart from factors which depend on the coefficients alone) form a basis for the ideal (f_i). The polynomial $d_n(x_n)$ is the resultant of the f_i considered as functions of $x_1, \cdots, x_{n-1}$. These results are applied to the case when the coefficients are taken from a field K. A system of such polynomials f_i^* is called reducible if the ideal they generate coincides with an ideal (F_i^*). The polynomials F_i^* enable the zeros of the ideal to be determined explicitly. A reducible system is called simple if all the zeros are simple, i.e., have a Jacobian not equal to 0. It is proved that to any n (general) f_i there correspond two polynomials in the coefficients whose nonvanishing, for a special choice of coefficients in K, is sufficient for the reducibility and for the simplicity of (f_i^*). The corresponding F_i^* are obtained by specialising the coefficients in F_i. A finer study is carried out when K is assumed to be ordered. Systems (f^*) of n and systems $((f^*))$ of $n+1$ polynomials are considered. They define mappings of the R_n (with coordinates in K) into the R_n and R_{n+1} by the correspondence: $x_i \to f_i^*(x_i)$. The index of (f^*) at a simple zero is the sign of the Jacobian there; the indicator of $((f^*))$ on a subset of R_n in which $((f^*))$ has no zeros and in which its projection $(\bar{f}^*) = (f_1^*, \cdots, f_n^*)$ has at most a finite number of simple zeros is given by the sum $\sum \text{index}(\bar{f}^*) \cdot \text{sgn } f_{n+1}^*$. It is shown how to replace the system $((f))$ by a simpler system $((F))$ such that $((f^*))$ and $((F^*))$ have the same indicator on any subset of R_n. From this system a further system (h) is obtained with the property that the indicator of $((f^*))$ is the sum of the indices of (h^*). The case $n=2$ is explained in detail. The results do not all apply when $n=1$. *O. Todd-Taussky.*

Fuchs, Ladislas. Domaines d'intégrité où tout idéal est quasi primaire. C. R. Acad. Sci. Paris 226, 1660–1662 (1948).

A quasiprimary ideal is one whose radical is prime. If R is an integral domain, it is proved that every ideal in R is quasiprimary if and only if any one of the following conditions holds. (1) Of any two prime ideals in R, one contains the other; (2) of any two elements of R, one divides some power of the other; (3) R is the valuation ring for a generalized valuation: the value group is partially ordered and of any two positive elements one is less than some multiple of the other. *I. S. Cohen* (Cambridge, Mass.).

Tamari, Dov. On a certain classification of rings and semigroups. Bull. Amer. Math. Soc. 54, 153–158 (1948).

The author solves a problem proposed by the reviewer [Ann. of Math. (2) 32, 463–477 (1931)]. Here a ring R is said to have right order of irregularity n_r if any n_r+1 but not n_r of its elements are linearly dependent with right coefficients in R. If R has divisors of zero, then $n_r = 0$, and when $n_r = 1$ the ring is regular, any two elements have a common left multiple and a quotient field exists. Each ring has a pair (n_r, n_l) of irregularity orders. The author proves that only rings with the following nine types of irregularity orders are possible: $(0, 0)$, $(0, 1)$, $(0, \infty)$, $(1, 0)$, $(\infty, 0)$, $(1, 1)$, $(1, \infty)$, $(\infty, 1)$, (∞, ∞). The remaining part of the paper is devoted to showing by examples that rings of these various types exist. This is mostly done by rings of polynomials in two variables with certain rules of interchange between the variables. *O. Ore* (New Haven, Conn.).

Barsotti, I. Il gruppo di Brauer delle algebre semplici di tipo I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 188–192 (1947).

A division algebra is of type 1 if every finite subset is contained in a central subalgebra of finite order; it is special if for every prime p there is a fixed upper bound to those degrees of subalgebras which are powers of p. The author extends the theory of the Brauer group to special division algebras of type 1. For the countable case this had been done by Köthe [Math. Ann. **105**, 15–39 (1931)]. The extension to the noncountable case is achieved by the use of a certain countable subalgebra called the characteristic subalgebra; the latter is defined and studied in a paper to appear elsewhere. *I. Kaplansky* (Princeton, N. J.).

Loonstra, F. Les systèmes hypercomplexes non commutatifs de deuxième ordre. Nederl. Akad. Wetensch., Proc. **51**, 220–223 = Indagationes Math. **10**, 91–94 (1948).

The author determines all noncommutative algebras of order two. The result is known [Cayley, Proc. London Math. Soc. (1) **15**, 185–197 (1884), and many later references; also ex. 13 on p. 218 of Birkhoff and MacLane, A Survey of Modern Algebra, Macmillan, New York, 1941]. *I. Kaplansky* (Princeton, N. J.).

Jacobson, N. The center of a Jordan ring. Bull. Amer. Math. Soc. **54**, 316–322 (1948).

The author takes an arbitrary associative ring $\mathfrak{A}$ and derives from it a Jordan ring $\mathfrak{A}_j$ and a Lie ring $\mathfrak{A}_l$, in which the products are $\{ab\}=ab+ba$, $[ab]=ab-ba$, respectively. He defines the center of a nonassociative ring to be the set of all elements of the ring which commute and associate with all elements of the ring. He shows that, if $\mathfrak{C}$, $\mathfrak{C}_l$ and $\mathfrak{C}_j$ are the centers of $\mathfrak{A}$, $\mathfrak{A}_l$ and $\mathfrak{A}_j$, respectively, then $\mathfrak{C}_l$ and $\mathfrak{C}_j$ are ordinary subrings of $\mathfrak{A}$ and, further, that $\mathfrak{C}\subseteq\mathfrak{C}_l\subseteq\mathfrak{C}_j$, the inequalities only holding if $\mathfrak{C}$ contains nilpotent elements (and not necessarily in this case). The elements of $\mathfrak{C}_j$ have the property that they commute with all commutators ab in $\mathfrak{A}$. Hence, if $\mathfrak{A}$ is semi-simple, the equality of $\mathfrak{C}$ and $\mathfrak{C}_j$ is a consequence of a result of Kaplansky [Duke Math. J. **14**, 521–525 (1947); these Rev. **9**, 172] to the effect that in a semi-simple ring elements commuting with all commutators belong to the center.

The above results are extended to an arbitrary Jordan subring $\mathfrak{U}$ of $\mathfrak{A}_j$, subject to the condition that the enveloping associative subring of $\mathfrak{U}$ is $\mathfrak{A}$ itself. If $\mathfrak{A}$ has no elements of additive order 2 and has no nilpotent elements in its center, the Jordan center of $\mathfrak{U}$ is shown to be the set of elements of $\mathfrak{U}$ commuting with every element of $\mathfrak{U}$, or indeed of $\mathfrak{A}$.

In a recent paper [Trans. Amer. Math. Soc. **59**, 524–555 (1946); these Rev. **8**, 63], Albert has defined the center of a Jordan subalgebra $\mathfrak{U}$ of a complete matrix algebra $\mathfrak{A}$ to be the set of all elements of $\mathfrak{U}$ which commute with all elements of $\mathfrak{U}$. The author shows, by an example, that, if $\mathfrak{A}$ is replaced by an arbitrary algebra, Albert's definition is not invariant under isomorphism of $\mathfrak{U}$. However, if $\mathfrak{A}$ is a complete matrix algebra and $\mathfrak{U}$ is simple, Albert's definition coincides with the one used in this paper. *D. Rees.*

THEORY OF GROUPS

Please read the corresponding note : p. 1277 of Math. Rev. 14.

Prachar, Karl. Zur Axiomatik der Gruppen. Akad. Wiss. Wien, S.-B. IIa. **155**, 97–102 (1947).

It is well known that the following axioms (i)–(iv) define a group: (i) if $a, b\in S$ then $ab\in S$, where ab is the unique "product" of the ordered pair a, b; (ii) $(ab)c=a(bc)$ for all $a, b, c\in S$; (iii) there exists at least one element $e\in S$ such that $ea=a$ for all $a\in S$; (iv) for every $a\in S$ there exists an element $a^{-1}\in S$ such that $a^{-1}a=e$. The author considers systems S satisfying (i), (ii), (iii) above and (iv'): for every $a\in S$ there exists an element $a^{-1}\in S$ such that $aa^{-1}=e$. If S has only a finite number of elements, the following result is obtained: S contains left units $e_1, \cdots, e_n$ such that $e_ia=a$ for all $a\in S$, $i=1, \cdots, n$. Let G_i be the subset of elements of S for which e_i is also a right unit; then the G_i are isomorphic groups with unit elements e_i, $i=1, \cdots, n$, and every element of S belongs to just one of the G_i. Set $G_1=G$, say, and $e_1=e$: then the elements of G_i may be written in the form ge_i, where $g\in G$, and the multiplication table of S may be constructed from that of G by means of the relations $(ge_i)(he_j)=(gh)e_j$ for all $g, h\in G$, $i, j=1, \cdots, n$. ~~The case where S has an infinity of elements is not discussed and is probably more difficult.~~ *S. A. Jennings* (Vancouver, B. C.).

Zassenhaus, Hans. Über einen Algorithmus zur Bestimmung der Raumgruppen. Comment. Math. Helv. **21**, 117–141 (1948).

This is an algebraic approach to the problem of enumerating the discrete groups of congruent transformations having finite fundamental regions. Such a group in Euclidean n-space is treated abstractly, as an extension of the free Abelian group with n generators. As the author remarks in a postscript, a similar procedure was worked out independently by J. J. Burckhardt [Die Bewegungsgruppen der Kristallographie, Birkhäuser, Basel, 1947; these Rev. **8**, 562]. *H. S. M. Coxeter* (Toronto, Ont.).

Piccard, Sophie. Note sur les bases du groupe symétrique. Comment. Math. Helv. **21**, 142–149 (1948).

The greater part of this paper is devoted to proving the following theorem. If A is any cycle of order n, and if A and any other substitution B together form a basis of the symmetric group S_n, then the n^2 substitutions A^iBA^j are distinct. It follows that for $n\geqq 4$ the number of such bases which contain A is a multiple of n^2. On the other hand the total number of such bases is also a multiple of n^2 provided n is odd and not less than 5; if n is even the theorem does not hold. In conclusion the author lists the bases of S_7 which are of this character. *G. de B. Robinson* (Toronto, Ont.).

Piccard, Sophie. Un théorème concernant le nombre total des bases d'un groupe d'ordre fini. Comment. Math. Helv. **21**, 150–153 (1948).

A finite group G of order N is said to possess a basis $B(b_1, \cdots, b_v)$ of order v provided B is a system of independent generators of G. If B is a basis so also is $B'=aBa^{-1}$ for any element a of G; if $B'\neq aBa^{-1}$ then B' is said to be independent of B. Denote by $B_1, \cdots, B_m$ a complete system of independent bases of G such that any other system is transformable into this one. The total number of bases of G may be written $\mathfrak{n}=(N/n_1)+\cdots+(N/n_m)$, so that if n is the least common multiple of the n_i, $\mathfrak{n}$ is a multiple of N/n. Moreover, $mN/(v!\mu)\leqq\mathfrak{n}\leqq mN$, where μ is the order of the centre of G. If G is Abelian, $n_i=n=N=\mu$, so that $\mathfrak{n}=m$, and we obtain no information concerning the factors of $\mathfrak{n}$. *G. de B. Robinson* (Toronto, Ont.).

* A complete discussion of this problem, including the infinite case, was given by H. B. Mann [Bull. Amer. Math. Soc. 50, p. 879-881 (1944), these Rev. 6, 147] and an extension to loops was announced by M. F. Smiley [Bull. Amer. Math. Soc. 53, 480 (1947)].

Benado, M. Séries canoniques principales du problème de Galois. Bull. Math. Soc. Roumaine Sci. **47**, 49–61 (1946).

[In the original "séries" appeared as "série."] Theorems on finite groups due to Fitting [Jber. Deutsch. Math. Verein. **48**, 77–141 (1938)], together with further theorems derived by similar methods, are applied to Galois theory. Sample theorems are as follows. (A) Every finite group G has a maximal normal subgroup g such that G/g is solvable. (B) All the semisimple normal subgroups of G are contained in the centralizer of the maximal normal subgroup of G. [Existence of the latter subgroup was proved by Fitting, loc. cit.] *G. Whaples* (Bloomington, Ind.).

Yeh, Yenchien. On prime power Abelian groups. Bull. Amer. Math. Soc. **54**, 323–327 (1948).

The author gives an explicit formula for the number of subgroups of any preassigned type of an Abelian p-group G of type $(k_1, \cdots, k_n)$. The method utilizes a formula of G. A. Miller for the number of cyclic subgroups of G. Two such subgroups of order p^λ are called isomers in G if a generator of one group is the product of an element in G of order less than p^λ and a generator of the other group. The proof by induction is based on the fact that the direct product of two cyclic subgroups of order p^λ is a group of order $p^{2\lambda}$ if and only if these subgroups are not isomers in G. *G. de B. Robinson* (Toronto, Ont.).

Černikov, S. N. On the theory of finite p-extensions of Abelian p-groups. Doklady Akad. Nauk SSSR (N.S.) **58**, 1287–1289 (1947). (Russian)

Die unendliche p-Gruppe $\mathfrak{G}$ enthalte einen Abelschen Normalteiler $\mathfrak{N}$ mit endlicher Faktorgruppe. Satz 1: Wenn $\mathfrak{G}$ endlichen Exponenten hat, so hat $\mathfrak{G}$ unendliches Zentrum $\mathfrak{z}$ und [Satz 2] $\mathfrak{G}$ ist nilpotent. Satz 3: Wenn umgekehrt $\mathfrak{G}$ nilpotent ist, dann hat $\mathfrak{G}/\mathfrak{z}$ endlichen Exponenten. Und folglich sind die Faktorgruppen der aufsteigenden Zentralreihe mit möglicher Ausnahme des ersten Gliedes direkte Produkte zyklischer Gruppen mit beschränkter p-Potenzordnung [Satz 4]. Satz 5: Wenn $\mathfrak{G}$ unendlich viele Elemente der Ordnung p besitzt, so gilt dasselbe für $\mathfrak{z}$. Satz 6: Eine unendliche p-Gruppe mit bis oben aufsteigender Zentralreihe endlicher Länge und Minimalbedingung für die Normalteiler von endlichem Index ist endliche Erweiterung eines direkten Produktes primärer Abelscher Gruppen vom Typus p^∞. Wenn unter den gleichen Voraussetzungen das Zentrum von $\mathfrak{G}$ nur endlich viele Elemente der Ordnung p enthält, so ist $\mathfrak{G}$ nilpotent [Satz 7]. Anstatt dessen kann auch die Minimalbedingung für Abelsche Normalteiler gefordert werden [Satz 8]. Satz 9: Jede lokal endliche p-Gruppe mit Minimalbedingung für Normalteiler ist nilpotent. Damit folgt auch die Minimalbedingung für beliebige Untergruppen. *H. Zassenhaus* (Hamburg).

Čunihin, S. A. On subgroups of relatively soluble groups. Doklady Akad. Nauk SSSR (N.S.) **58**, 1295–1296 (1947). (Russian)

Verfasser nennt eine endliche Gruppe auflösbar (bzw. stark auflösbar) bezüglich der Primzahl p, wenn jeder Kompositionsfaktor (bzw. jeder Faktor der Hauptreihe) entweder p ist oder zu p teilerfremd ist. Eine Folgerung des Satzes von Burnside über die Auflösbarkeit der Gruppen der Ordnung $p^a q^b$ ist Satz 1: Die Anzahl der Primzahlen, bezüglich deren eine nicht auflösbare Gruppe nicht auflösbar ist, übersteigt 2. Auf ähnliche Weise erscheinen die Sätze 2 und 3 als Verallgemeinerung der Sätze von P. Hall [J. Lond. Math. Soc. **3**, 98–105 (1928)]. *H. Zassenhaus.*

Čunihin, S. A. On Π-separable groups. Doklady Akad. Nauk SSSR (N.S.) **59**, 443–445 (1948). (Russian)

Durch vollständige Induktion nach der Gruppenordnung wird in Verallgemeinerung eines Satzes von P. Hall [J. London Math. Soc. **3**, 98–105 (1928)] und früherer Arbeiten vom Verfasser [siehe vorstehendes Referat und C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 477–480 (1947); diese Rev. **9**, 6] bewiesen: Wenn sich die Ordnung g der Gruppe in zwei zu einander teilerfremde Faktoren m und g/m zerlegen lässt, sodass jeder Kompositionsfaktor der Gruppe mit m höchstens eine Primzahlpotenz gemein hat, dann gibt es Untergruppen der Ordnung m und alle Untergruppen der Ordnung m sind unter der ganzen Gruppe konjugiert. *H. Zassenhaus* (Hamburg).

Szekeres, G. Countable Abelian groups without torsion. Duke Math. J. **15**, 293–306 (1948).

Let $C_1, \cdots, C_k$ be elements of an Abelian group $\mathfrak{A}$ without elements of finite order. The author considers congruences $\alpha_1 C_1 + \cdots + \alpha_k C_k \equiv 0 \ (p^\infty)$, where $\alpha_1, \cdots, \alpha_k$ are p-adic integers; and if $\alpha_i^{(n)}$ denotes the nth convergent of α_i, the congruence signifies that for all n, $\sum_i \alpha_i^{(n)} C_i \equiv p^n \cdot \mathfrak{A}$. This leads to a theory of linear dependence modulo p^∞. Now suppose $\mathfrak{A}$ to be countable and to have the rational base $A_1, A_2, \cdots, A_i, \cdots$. Using this theory, the author constructs a set of elements of $\mathfrak{A}$ which generate $\mathfrak{A}$ subject to relations of a specified type. Into these relations there enter certain parameters, which are rational or p-adic integers; and subject to certain conditions the author shows that every set of parameters corresponds to one and only one group, though of course different sets of parameters may give rise to the same group. That is to say, the author's construction gives a semi-normal form for countable Abelian groups without torsion. Finally the author shows how certain group-invariants can be obtained from these parameters, including a generalisation of Kurosch's "reduced rank" [Ann. of Math. (2) **38**, 175–203 (1937)], and obtains a necessary condition that a group should be completely reducible, and a necessary and sufficient condition for it to be a free Abelian group. *G. Higman* (Manchester).

Szekeres, G. On a certain class of metabelian groups. Ann. of Math. (2) **49**, 43–52 (1948).

The author determines all extensions $\mathfrak{G}$ of an elementary Abelian group $\mathfrak{A}$ of order p^λ and type $(1, 1, \cdots, 1)$ by a cyclic group of order n, $\mathfrak{B} = \mathfrak{G}/\mathfrak{A}$. The method is more explicit than previous work [cf., e.g., R. Baer, Math. Z. **38**, 375–416 (1934); M. Hall, same Ann. (2) **39**, 220–234 (1938)] and is based on a suggestion by A. Scholz [S.-B. Heidelberger Akad. Wiss. **2**, 17–22 (1933)]: the factor-group $\mathfrak{B}$ is considered as operator-domain for $\mathfrak{A}$ and is extended in the usual way to form a ring $R(\mathfrak{B})$, which is a subring of the complete ring of endomorphisms of $\mathfrak{A}$ (i.e., in the present case the ring of matrices of order h with integer coefficients reduced mod p). The structure of $\mathfrak{A}$ as an $R(\mathfrak{B})$-module is then investigated.

The author formulates the relevant parts of the extension theory for groups in a mixture of additive and multiplicative notation which is suitable for the present problem. Sets of integers are obtained as "invariants" of the groups under consideration, and their invariant character is proved. The author obtains as an application of the preceding results all soluble groups $\mathfrak{G}$ which have the property that to any element $A \neq E$ another element B can be found such that A and B generate $\mathfrak{G}$. [The alternating group has this property for $n \neq 4$; cf. S. Piccard, Comment. Math. Helv. **11**,

1–8 (1938).] The Abelian groups of this class are either cyclic or of order p^2 and type (1, 1); to any given prime p and to any $m>1$ with $(m, p)=1$ there exists one and only one non-Abelian soluble group G of order $p^e \cdot m$ (where e is the index to which p belongs mod m) with the required property. *K. A. Hirsch* (Newcastle-upon-Tyne).

Mal'cev, A. I. On groups of finite rank. Mat. Sbornik N.S. **22(64)**, 351–352 (1948). (Russian)

The concept of rank [Prüfer, Math. Z. **20**, 165–187 (1924)] is extended from commutative to noncommutative groups in two ways. The general rank R of a group $\mathfrak{G}$ is defined to be the least positive integer (or ∞) with the property that every finite subset of $\mathfrak{G}$ lies in a subgroup of $\mathfrak{G}$ with R generators. The special rank r is defined by replacing "lies in" by "generates." In the non-Abelian case these may differ, as in the free group of n generators, where $R=n$ and $r=\infty$. The author shows that not all groups of finite general rank are at most countable. However, if $\mathfrak{G}$ is isomorphic to a group of n by n matrices with entries in a field K, then $\mathfrak{G}$ is at most countable. He proves this by appending to the simple field P of K all the entries of all the matrices which represent $\mathfrak{G}$ to obtain a new field Q which he then closes algebraically, using a result of Steinitz [J. Reine Angew. Math. **137**, 167–309 (1910)]. From this it turns out that Q, and consequently $\mathfrak{G}$, are countable. A consequence of this result is that every locally free group [Kurosch, C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 99–101 (1939); these Rev. **2**, 127] of finite general rank is countable. *F. Haimo* (St. Louis, Mo.).

Kontorovič, P. G. Groups with a basis of partition. III. Mat. Sbornik N.S. **22(64)**, 79–100 (1948). (Russian)

[Parts I and II appeared in Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 56–70 (1943); **19(61)**, 287–308 (1946); these Rev. **5**, 144; **8**, 437.] Part III is chiefly concerned with R-groups and their isolated subgroups. The partition of a group [cf. part II for definitions] is complete if the components are locally cyclic. A group possesses a complete partition if and only if every noncyclic subgroup with finitely many generators has a partition. A free product of groups is partitioned; the components of the partition include the free factors and all their conjugates. If the components of partition of a group are invariant subgroups, the group is Abelian. If the center Z of a group with partition is not 1, then Z is a p-group of type p or is a group without torsion (i.e., has no element of finite order except 1). If the center of a completely partitioned group is not 1, then the group is periodic or without torsion. A completely partitioned group without torsion is called an R-group. Since an Abelian group without torsion is a special case of an R-group, the present paper is closely related to previous work by Baer [Duke Math. J. **3**, 68–122 (1937)]. A group without torsion is an R-group if and only if the centralizer of every element is an isolated subgroup. Hence a group is an R-group if and only if it obeys the law: the extraction of a root of an element in the group is unique whenever the extraction is possible. The class of R-groups includes free groups, locally free groups, free products of R-groups, subgroups of R-groups, groups without torsion in which every maximal Abelian subgroup is isolated, groups coverable by invariant R-groups. In an arbitrary group, the intersection of all isolated subgroups containing a given subgroup H is called the isolator of H; basic properties of isolators are studied. Results connecting R-groups and complete groups are obtained. Elements in an R-group may be classified according to genus, with the aid of the Steinitz generalized numbers [cf. Baer, loc. cit.]. An element a in the set A is called primitive in A if, for each integer m, the equation $x^m=a$ is solvable provided $y^m \varepsilon A$ is solvable. An isolated invariant subgroup F in an R-group G with locally cyclic quotient group is a semidirect factor of G if and only if at least one coset other than F has a primitive element; when F is a semidirect factor, every coset except F has a primitive element. Direct products of R-groups are investigated. The direct decomposition of an R-group is unique to within isomorphisms, provided elements in the same factor of decomposition have the same genus while elements in different factors have different genera. The factor group of a group without torsion G modulo its invariant isolated subgroup F is an R-group if and only if the commutator quotient $F \div g$ is an isolated subgroup for every element $g \varepsilon G$. A group without torsion is an R-group if and only if the factor group modulo the center is an R-group. Also discussed are central series, especially those in which every term is an isolated subgroup. *R. A. Good* (College Park, Md.).

Golovin, O. N. On associative operations on a set of groups. Doklady Akad. Nauk SSSR (N.S.) **58**, 1257–1260 (1947). (Russian)

Mit $\overline{\mathfrak{U}}$ möge der von der Untergruppe $\mathfrak{U}$ der Gruppe $\mathfrak{G}$ erzeugte Normalteiler der Gruppe $\mathfrak{G}$ bezeichnet werden. Die Gruppe $\mathfrak{G}$ heisst reguläres Produkt ihrer Untergruppen $\mathfrak{A}$ und $\mathfrak{B}$, wenn $\mathfrak{G}=\langle \mathfrak{A}, \mathfrak{B} \rangle$, $\mathfrak{A} \cap \mathfrak{B}=\mathfrak{A} \cap \overline{\mathfrak{B}}=1$. Z.B. sind die freien Produkte, desgleichen die direkten Produkte stets regulär. Satz 1: ein reguläres Produkt ist genau dann direkt, wenn einer der Faktoren Normalteiler ist. [Anm. des Ref.: stets ist die aus den Kommutatoren $(A, B)=ABA^{-1}B^{-1}$ der Elemente A aus $\mathfrak{A}$, B aus $\mathfrak{B}$ erzeugte Untergruppe Normalteiler des Erzeugnisses $\langle \mathfrak{A}, \mathfrak{B} \rangle$. Wenn $\mathfrak{G}$ reguläres Produkt von $\mathfrak{A}$ und $\mathfrak{B}$ ist, so ist $\mathfrak{G}/(\mathfrak{A}, \mathfrak{B}) \simeq \mathfrak{A} \times \mathfrak{B}$.] Satz 2: $\mathfrak{G}$ ist genau dann reguläres Produkt der beiden Untergruppen $\mathfrak{A}$ und $\mathfrak{B}$, wenn für jedes Element aus $\mathfrak{G}$ genau eine Darstellung in der Form abu mit a aus $\mathfrak{A}$, b aus $\mathfrak{B}$, u aus $(\mathfrak{A}, \mathfrak{B})$ besteht. Hierbei kann in der Formulierung von Satz 2 auch irgend eine andere Reihenfolge der drei Faktoren a, b, u festgesetzt werden. Die Komponenten a, b sind unabhängig von der festgesetzten Reihenfolge. Satz 3: Jedes reguläre Produkt der Faktoren $\mathfrak{A}, \mathfrak{B}$ ist isomorph zu einer Faktorgruppe des freien Produktes $\mathfrak{F}=\mathfrak{A} * \mathfrak{B}$ nach einem Normalteiler von $\mathfrak{F}$, der in $(\mathfrak{A}, \mathfrak{B})_{\mathfrak{F}}$ enthalten ist.

Gemäss Baer [Trans. Amer. Math. Soc. **58**, 348–389 (1945); diese Rev. **7**, 372] wird die absteigende Zentralreihe der Gruppe $\mathfrak{G}$ bezüglich eines Normalteilers $\mathfrak{N}$ rekursiv erklärt: ${}_0\mathfrak{N}=\mathfrak{N}$, ${}_k\mathfrak{N}=(\mathfrak{G}, {}_{k-1}\mathfrak{N})$. Das k-te Zentralprodukt der beiden Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ wird nun durch die Formel $\mathfrak{A}(k)\mathfrak{B}=\mathfrak{A} * \mathfrak{B}/{}_k(\mathfrak{A}, \mathfrak{B})_{\mathfrak{F}}$ erklärt. Man zeigt, dass es sich dabei, nach geeigneter Identifizierung von $\mathfrak{A}$ bzw. $\mathfrak{B}$ mit Untergruppen von $\mathfrak{A}(k)\mathfrak{B}$ um ein reguläres Produkt von $\mathfrak{A}$ und $\mathfrak{B}$ handelt. Ferner ist $\mathfrak{A} \times \mathfrak{B} \simeq \mathfrak{A}(0)\mathfrak{B}$, $\mathfrak{A}(k)\mathfrak{B} \simeq \mathfrak{B}(k)\mathfrak{A}$, Satz 4 und 5:

$$(\mathfrak{A}(k)\mathfrak{B})(k)\mathfrak{C} \simeq \mathfrak{A}(k)(\mathfrak{B}(k)\mathfrak{C}) \simeq \mathfrak{A} * \mathfrak{B} * \mathfrak{C}/{}_k\langle \overline{(\mathfrak{A}, \mathfrak{B})(\mathfrak{B}, \mathfrak{C})(\mathfrak{C}, \mathfrak{A})} \rangle.$$

Das erste Zentralprodukt ist bereits von F. W. Levi [J. Indian Math. Soc. (N.S.) **8**, 78–91 (1944); diese Rev. **7**, 113] behandelt worden und dort wurde bereits gezeigt, dass $(\mathfrak{A}, \mathfrak{B})_{\mathfrak{F}^{(1)}} \simeq (\mathfrak{A}/(\mathfrak{A}, \mathfrak{A}), \mathfrak{B}/(\mathfrak{B}, \mathfrak{B}))_{\mathfrak{F}^{(1)}}$, und ferner $(\mathfrak{A}_1 \times \mathfrak{A}_2, \mathfrak{B})_{\mathfrak{F}^{(1)}} = (\mathfrak{A}_1, \mathfrak{B})_{\mathfrak{F}^{(1)}} \times (\mathfrak{A}_2, \mathfrak{B})_{\mathfrak{F}^{(1)}}$ ist. Dabei wird allgemein mit $(\mathfrak{A}, \mathfrak{B})_{\mathfrak{F}^{(k)}}$ der in $\mathfrak{A}(k)\mathfrak{B}$ ge-

bildete k-te Zentralkommutant ${}_k(\mathfrak{A}, \mathfrak{B})$ bezeichnet, sodass $(\mathfrak{A}, \mathfrak{B})_\varrho^{(\alpha)} \simeq (\mathfrak{A}, \mathfrak{B})_\varrho / {}_k(\mathfrak{A}, \mathfrak{B})_\varrho$ ist. Schliesslich wird die dem k-ten Zentralprodukt entsprechende Bildung für Produkte von unendlich vielen Faktoren $\mathfrak{A}_\alpha$ als die Faktorgruppe des freien Produktes der Gruppen $\mathfrak{A}_\alpha$ nach ${}_k(\overline{\prod_{\alpha<\beta}(\mathfrak{A}_\alpha, \mathfrak{A}_\beta)})$ erklärt. Durch transfinite Induktion ist zu beweisen, dass diese Produktbildung kommutativ und [Satz 6] assoziativ im erweiterten Sinne ist. *H. Zassenhaus* (Hamburg).

Dieudonné, Jean. Sur les automorphismes des groupes classiques. C. R. Acad. Sci. Paris **225**, 914–915 (1947).

The following results are announced by the author. (i) He has determined the group of all automorphisms of a general linear group over any field, commutative or not, which is an extension of a theorem due to Schreier and van der Waerden [Abh. Math. Sem. Hamburgischen Univ. **6**, 303–322 (1928)]. (ii) He has determined the group of all automorphisms of the symplectic group of $2m(\neq 4)$ rowed matrices over the commutative field with more than 2 elements [which was also obtained by the reviewer]. (iii) He has determined the group of all automorphisms of the orthogonal group of index $\nu \geqq 1$ of $n(\geqq 4)$ rowed matrices over any commutative field with characteristic not equal to 2. No proofs are indicated. [The reviewer has obtained some similar results: abstract in Bull. Amer. Math. Soc. **53**, 48 (1947).]

L.-K. Hua (Urbana, Ill.).

Dieudonné, Jean. Sur les automorphismes du groupe unitaire. C. R. Acad. Sci. Paris **225**, 975–976 (1947).

Let K be a quadratic extension of a commutative field K_0 of characteristic not equal to 2, $\lambda \to \bar{\lambda}$ the unique automorphism of K, different from the identity, leaving the elements of K_0 invariant. Let E be a vector space of dimension n on K and let $f(x, y)$ be a bilinear Hermitian form [i.e., $f(\lambda x, y) = \lambda f(x, y)$, $f(y, x) = \overline{f(x, y)}$]; the unitary group $U_n(k, f)$ is the subgroup of the general linear group over K satisfying the following condition: if $u \in U_n(K, f)$, then $f(ux, uy) = f(x, y)$ identically. Suppose f to be of index $\nu \geqq 1$. The author states that he has determined the group of all automorphisms of $U_n(K, f)(n \geqq 4)$. He indicates that the proof of the results of this and the preceding article depend on his study of the structure of the classical groups [see the following review] and a method of G. W. Mackey [Ann. of Math. (2) **43**, 224–260 (1942); these Rev. **4**, 12].

L.-K. Hua (Urbana, Ill.).

Dieudonné, Jean. Sur les groupes classiques. Actualités Sci. Ind., no. 1040 = Publ. Inst. Math. Univ. Strasbourg (N.S.) no. 1 (1945). Hermann et Cie., Paris, 1948. iii+82 pp.

The term classical groups covers for the author [as it did for the reviewer, The Classical Groups, Princeton University Press, 1939; these Rev. **1**, 42] the group of all linear transformations in n variables and those of its subgroups that leave certain nondegenerate forms invariant, namely either a quadratic or a Hermitian or a skew-symmetric bilinear form (orthogonal, unitary, symplectic groups). In spite of this common nomenclature there is practically no overlapping whatsoever between the two books. For Dieudonné deals with the structure of the groups, not with their representations and invariants, and he makes no assumption about the underlying number field K; indeed, fields of prime characteristic provide the more interesting examples for his general results. After C. Jordan in his Traité des Substitutions considered linear groups in the field of residues modulo a prime number and L. E. Dickson for all finite fields, Dieudonné now turns to a systematic study of the general case, borrowing of course to some extent from his predecessors, but greatly simplifying and unifying their methods and adding new methodical ideas. E. Witt's investigation of quadratic and Hermitian forms in arbitrary fields serves as a starting point for the analysis of the orthogonal and unitary groups. In particular, Witt's general notion of index is of fundamental importance. The author disregards the full linear group; of the other groups the symplectic one proves to be the simplest in nature.

Given a nondegenerate skew-symmetric bilinear form $f(x, y) = (x, y)$ in an n-dimensional vector space E, the values of which are supposed to be numbers of a given field K, the symplectic group $Sp_n(K)$ consists of all linear transformations leaving this form invariant. The dimension is necessarily even, $n = 2m$, and for f there exists a universal normal form. For any linear subspace V of E we have the conjugate subspace V^* to which a vector x belongs if $(x, y) = 0$ for every $y \in V$. Then V is called isotropic if $V \cap V^*$ is not empty; it is called totally isotropic if $(x, y) = 0$ for $x, y \in V$. The maximum dimension of a totally isotropic subspace is m. A symplectic transformation is called singular if it leaves invariant all the vectors of a totally isotropic subspace of maximum dimension m. A transformation $x \to x - \lambda(x, a)a$, in which a is a fixed vector and λ a fixed number, is a symplectic transvection. With these notions the following propositions hold.

(1) Every totally isotropic subspace is contained in such a subspace of maximum dimension m. Two totally isotropic subspaces of the same dimension are symplectically equivalent. (2) Any two subspaces V_1, V_2 of the same dimension are symplectically equivalent provided the two forms arising from the fundamental form (x, y) by restricting the arguments to V_1 or V_2, respectively, are of equal rank. (3) Let u be an involutorial symplectic transformation. If the characteristic of K is not 2, then there exists a nonisotropic subspace V such that u is the identity in V and the transformation $x \to -x$ in the conjugate subspace V^*. If, however, K is of characteristic 2, then u is singular. (4) Each symplectic transformation is the product of (at most $2n$) symplectic transvections.

Let F_q denote the finite field consisting of q elements. About the structure of the symplectic group the author proves the following theorem, which goes beyond a result obtained by Dickson in that it includes imperfect fields of characteristic 2: apart from the groups $Sp_2(F_2)$, $Sp_2(F_3)$ and $Sp_4(F_2)$ no $Sp_n(K)$ contains an invariant subgroup different from itself and not contained in its center.

For the orthogonal group $O_n(K, f)$ which is based on a given nondegenerate quadratic form f with values in a field K of characteristic not 2 there holds a proposition similar to (1). Only m is now to be replaced by the index ν, which is the maximum dimension of totally isotropic subspaces, $2\nu \leqq n$. Proposition (2) holds under the more incisive hypothesis that the restrictions of f to V_1 and V_2 are equivalent. This fact limited to nonisotropic V_1, V_2 had been one of Witt's main results. Proposition (3) carries over unchanged. In analogy to proposition (4) one finds: (4*) each orthogonal transformation is a product of at most n symmetries, a symmetry leaving all the vectors of a nonisotropic $(n-1)$-dimensional subspace V invariant and changing the vectors x in the 1-dimensional conjugate V^* into $-x$.

The orthogonal transformations of determinant $+1$, that form the group $O_n^+(K, f)$, are called rotations. A rotation

leaving the vectors of an $(n-2)$-dimensional nonisotropic V invariant is called a plane rotation and V^* its plane. A renversement is an involutorial plane rotation. (4**) If $n>2$ then each rotation is a product of renversements. (5*) The commutator group $\Omega_n(K, f)$ of $O_n(K, f)$ is generated by the products of two conjugate symmetries (conjugate in the group-theoretic sense). It is also generated by the squares of the elements of $O_n(K, f)$.

General results about the structure of the group can be proved if the index $\nu \geqq 1$. A nonisotropic plane containing (two) isotropic straight lines is called hyperbolic, and a plane rotation whose plane is hyperbolic a hyperbolic rotation. (6*) For $n \geqq 2$ and $\nu \geqq 1$ every rotation is a product of hyperbolic rotations. (7*) If $n>2$ and $\nu \geqq 1$ then the (Abelian) factor group $O_n^+(K, f)/\Omega_n(K, f)$ is isomorphic to the subgroup of the factor group K^*/Q, where K^* is the multiplicative group of the numbers not equal to 0 of K and Q the group of the squares of the elements of K^*.

Let Z_n be the center of O_n. Then $\Omega_n/(\Omega_n \cap Z_n)$ is isomorphic to the commutator group of the "projective" orthogonal group O_n/Z_n. The decisive structural theorem about the orthogonal group is as follows. For $n \geqq 5$ and $\nu \geqq 1$ the group $\Omega_n/(\Omega_n \cap Z_n)$ is simple; or more explicitly: if G is any subgroup of O_n containing Ω_n then every invariant subgroup of G not contained in Z_n contains Ω_n (provided $n \geqq 5$ and $\nu \geqq 1$). In the case $\nu=0$ no such general results are obtainable. Examples show that then the structure of the group depends in a very essential manner on the structure of the field K.

The orthogonal group in a field K of characteristic 2 requires a special study. Here $g(x)$ is a quadratic form if $g(\lambda x+\mu y)=\lambda^2 g(x)+\mu^2 g(y)+\lambda\mu f(x, y)$ where x, y are arbitrary vectors, λ, μ arbitrary numbers and $f(x, y)$ is a bilinear form, that is necessarily symmetric and hence antisymmetric. Let $2p$ be its rank and introduce the "defect" $d=n-2p$. If the defect is zero, the antisymmetric form $f(x, y)=(x, y)$ is nondegenerate, and every orthogonal transformation is at the same time symplectic. The results in this case are none too different from the ones obtained for a field of characteristic not 2. But instead of symmetries, transvections, namely symplectic transvections that are orthogonal, reappear. Let $q \leqq n-2p$ be the dimensionality of the subspace E_0 to which a vector x belongs if $g(x)=0$ and $f(x, y)=0$ for every y. Even if the defect $d>0$, it is assumed that this q vanishes; forms $g(x)$ which are not regular in this sense are not studied by the author. This case $d>0$, $q=0$ still permits of as complete an analysis as the previous cases.

The problem of the unitary group may be interpreted in two different ways, a commutative and a noncommutative one. In the commutative interpretation K is assumed to be a separable extension of degree 2 over a given field K_0, and the definition of a Hermitian bilinear form $f(x, y)$ makes use of the nonidentical automorphism $\xi \to \bar{\xi}$ of K/K_0. In the noncommutative interpretation, K is a division algebra with the center K_0, possessing an involutorial anti-automorphism. (As one knows, the degree of K/K_0 then equals 4 and, if the characteristic of K_0 is not 2, K is a generalized quaternion algebra.) One must distinguish between left and right vector spaces according to the position of the numerical factors which vectors are capable of taking on. Suppose that x and y vary over a right vector space. The behavior of the fundamental form f under multiplication of its arguments by numbers λ is to be defined by $f(x\lambda, y)=\bar{\lambda}f(x, y)$, $f(x, y\lambda)=f(x, y)\lambda$. The results obtained for these two sorts of unitary groups are analogous to those mentioned for the symplectic and orthogonal groups. *H. Weyl.*

Gel'fand, I. M., and Naĭmark, M. A. Unitary representations of the Lorentz group. Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 411–504 (1947). (Russian)

Basic results in harmonic analysis are extended from the case of locally compact Abelian and compact groups to the case of the complex unimodular group G in two dimensions (i.e., the multiplicative group of 2×2 complex matrices of determinant 1; the quotient of this simply connected group modulo its two-element center is the Lorentz group), and the (strongly) continuous irreducible unitary representations of G on Hilbert spaces are explicitly determined. In particular, analogues of the Plancherel and Stone theorems concerning the additive group of the reals are formulated and proved for G; and by virtue of the mutual correspondence between unitary representations of and positive definite functions on a group [same authors, Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 197–213 (1943); these Rev. **5**, 147], the latter analogue implies an analogue to the Herglotz-Bochner theorem. The group G is the (only within local isomorphism) noncompact complex semi-simple Lie group of lowest dimension and it is stated that the methods of the present paper can be applied to arbitrary complex semi-simple Lie groups [cf. the authors, Mat. Sbornik N.S. **21(63)**, 405–434 (1947); these Rev. **9**, 328]. Many of the proofs are along classical lines, much use being made of the Plancherel theorem, approximation by smooth functions on the group and related manifolds, and of bounds for norms of smooth functions by integrals involving transforms of the functions. The continuous irreducible unitary representations of G have also been found by Harish-Chandra [Proc. Roy. Soc. London. Ser. A. **189**, 372–401 (1947); these Rev. **9**, 132] in infinitesimal form, and by Bargmann [cf. Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 133] in the same form as in the present paper, though by infinitesimal methods (i.e., the study of the representations of the Lie algebra of G), which are very different from those employed by the present authors; that the set of representations obtained constitutes all such representations is, however, proved completely, for the first time, in the present paper.

The continuous irreducible unitary representations of G fall into two distinct classes, designated as the principal and the complementary series, the first of which consists of those irreducible representations which are "contained in" the regular representation of G on $L_2(G)$ (where $L_p(X)$ denotes the space of pth power integrable functions relative to the invariant measure on X). To obtain the principal series, G is represented in the natural fashion on $L_2(G/H)$, where H is the subgroup of G consisting of the matrices in G which have zero in the lower left corner, and the G-invariant measure on the coset space G/H is used. The resulting representation is then decomposed into a direct integral of irreducible representations.

First a matrix factorization theorem is used to regard elements f of $L_2(G/H)$ as functions of a complex variable z and a variable which ranges over the maximal Abelian subgroup D of diagonal matrices in G. Then essentially the Fourier transform of f as a function on D is taken, thereby arriving at a function $F_\chi(z)$, where χ varies over the character group D^* of D. Next it is observed that under the action of an element a of G, f maps into a function f' in such a way that f_χ' depends only on f_χ and a. Finally, for

each fixed χ the transformation from f_χ to f_χ' determines a representation of G which, with an appropriate measure on the complex plane, is unitary on the corresponding L_2, and continuous and irreducible; these are the representations of the principal series. Two elements χ_1 and χ_2 of D^* correspond to unitarily equivalent representations if and only if either $\chi_1=\chi_2$ or $\chi_1=\chi_2^{-1}$; or more significantly, if and only if there is an inner automorphism of G which leaves D fixed, and such that the automorphism thereby induced on D^* takes χ_1 into χ_2. Thus to each point of the space E obtained from D^* by identifying inverse elements there corresponds a unique unitary-equivalence class of representations in the principal series. In a somewhat similar fashion the complementary series is obtained. All the continuous irreducible unitary representations of G are obtained from point transformations on the complex plane, a fact which permits the formation of a very concrete type of direct integral of such representations.

For any continuous irreducible unitary representation U of G, the operator $\int_G U(a)f(a)da$, da being the element of Haar measure, is in the Hilbert-Schmidt class when f is continuous and vanishes outside a compact set, and completely continuous when $f\epsilon L_1(G)$. All such representations (except the trivial one) are infinite-dimensional, so that $U(a)$ will never be in the Hilbert-Schmidt class, but a formal trace $\tau_U(a)$ can be defined such that $\mathrm{tr}\,(\int_G U(a)f(a)da)=\int_G \tau_U(a)f(a)da$ for f continuous and vanishing outside a compact set, so that there is an analogue for the characters of irreducible representations of Abelian and compact groups. The analogue to Plancherel's theorem can be stated as follows, for simplicity not in most general form: if $f\epsilon L_1(G)\cap L_2(G)$, then $\int_G |f(a)|^2da=\int_E \mathrm{tr}\,(F_\epsilon{}^*F_\epsilon)d\mu(\epsilon)$, where $F_\epsilon=\int U(a)f(a)da$, U being a representation in the unitary-equivalence class ϵ (F_ϵ is a generalization of the Fourier transform), μ is a certain measure on E, and the asterisk denotes the adjoint. In a more refined form the analogue to Plancherel's theorem assigns to every element $f\epsilon L_2(G)$ a function $K(z_1, z_2, \epsilon)$ on $S=C\times C\times E$, C being the complex plane, which is the kernel in a representation of F_ϵ as an integral operator, and asserts that, for a certain measure on S, the mapping $f\to K$ is isometric and onto from $L_2(G)$ to $L_2(S)$. Another form of essentially the same theorem is the result that the regular representation of G is a direct integral relative to a stated measure of the representations in the principal series.

Finally it is shown that every continuous unitary representation of G is a direct integral of representations of the principal and complementary series. A continuous unitary representation of G gives rise to positive definite functions on G which give rise to linear functionals on $L_1(G)$ which are positive, in the sense of being nonnegative on the convolution of an element of $L_1(G)$ with its adjoint. It is sufficient in the present case to consider a certain commutative subalgebra of $L_1(G)$ which is isomorphic to an algebra of smooth functions of a complex variable, relative to a convolution-type product. The elementary positive linear functions on this algebra, i.e., those which are not linear combinations with positive coefficients of two other positive linear functions, are determined explicitly, and it is shown that every positive linear function is an integral, relative to a measure on the space of elementary functions, of the elementary functions. From this the above decomposition theorem for representations follows. *I. E. Segal.*

Gel'fand, I. M., and Yaglom, A. M. General Lorentz invariant equations and infinite representations of the Lorentz group. Doklady Akad. Nauk SSSR (N.S.) **59**, 655–658 (1948). (Russian)

The authors study equations of the form

$$(1)\qquad \sum_{j=0}^{3} L^j\partial\psi/\partial x^j+i\kappa\psi=0,$$

where κ is a real constant, $(x^0, x^1, x^2, x^3)=(ct, x, y, z)$, and the function ψ assumes values in a finite or infinite dimensional linear space R. The L^j are operators (matrices) on R. The equation (1) is called Lorentz invariant if for every Lorentz transformation $l_j{}^i$ there exists an operator S such that $L^i=\sum_j l_j{}^i SL^jS^{-1}$. Consequently these operators define a representation of the Lorentz group. For a finite dimensional space R, the general form of the matrices L^j has been discussed by H. J. Bhabha [Rev. Modern Physics **17**, 200–216 (1945); these Rev. **7**, 272] and by R. Potier [C. R. Acad. Sci. Paris **222**, 638–640 (1946); these Rev. **7**, 447]. In this paper the authors determine the L^j both for finite and for infinite dimensional representations of the Lorentz group. Assuming the representation S to be a direct sum of irreducible finite dimensional or unitary infinite dimensional representations, they obtain the operators L^j from their commutators with the infinitesimal generators $I^{ij}(=-I^{ji})$ of the representation S in an explicit form. [The infinite dimensional unitary representations have been determined by I. M. Gelfand and M. A. Neumark, cf. Acad. Sci. USSR. J. Phys. **10**, 93–94 (1946); these Rev. **8**, 132, and the paper reviewed above. Cf. also Harish-Chandra, Proc. Roy. Soc. London. Ser. A. **189**, 372–401 (1947); V. Bargmann, Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 132, 133.] It is always possible to define a Hermitian form (ψ, ψ) invariant under the transformations S, such that $(L^0\psi, \psi)=(\psi, L^0\psi)$. The equations (1) may then be derived from a variational principle with the Lagrangian

$$\mathfrak{L}=\kappa(\psi, \psi)+(2i)^{-1}\sum_j\{(L^j\partial\psi/\partial x^j, \psi)-(\psi, L^j\partial\psi/\partial x^j)\}.$$

If S contains only a finite number of infinite dimensional representations, the resulting masses form a sequence which converges to zero. [Reviewer's note. The special case where S itself is an irreducible unitary representation has been treated by E. Majorana, Nuovo Cimento (N.S.) **9**, 335–344 (1932).] *V. Bargmann* (Princeton, N. J.).

Montgomery, Deane. Connected one-dimensional groups. Ann. of Math. (2) **49**, 110–117 (1948).

L'auteur démontre qu'un groupe connexe et localement compact (mais non-compact) à une dimension est isomorphe au groupe additif des nombres réels. Il suppose d'abord que G ne contienne aucun sous-groupe isomorphe au groupe réel. Alors il peut trouver un voisinage V de 1 tel que tous les itérés d'éléments de V soient contenus dans un voisinage U donné d'avance. Cette possibilité est refuté avec des méthodes d'homologie. Ayant trouvé un sous-groupe H isomorphe au groupe réel, on peut construire une section transversale de la partition de G d'après les classes aH, ce qui est impossible dès que la dimension de G est égale à 1. *H. Freudenthal* (Utrecht).

Montgomery, Deane. Analytic parameters in three-dimensional groups. Ann. of Math. (2) **49**, 118–131 (1948).

L'auteur résout le cinquième problème de Hilbert pour les groupes connexes totalement euclidiens au dimension trois. D'après des théorèmes du même auteur on sait que

le groupe G à trois dimensions doit posséder un sous-groupe fermé et connexe de deux dimensions ou d'une dimension, et l'on connaît la structure des groupes à une dimension. Une analyse qui fait usage des résultats de Brouwer sur les groupes de variétés à une et deux dimensions et des résultats de Hilbert concernant la caractérisation des groupes des géométries euclidienne et non-euclidienne, conduit au théorème énoncé. *H. Freudenthal* (Utrecht).

Vilenkin, N. Ya. On the theory of weakly separable groups. Mat. Sbornik N.S. **22(64)**, 135–177 (1948). (Russian)

The author introduced the concept of a weakly separable (Abelian) group in an earlier paper [Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 311–340 (1946); these Rev. **8**, 312]. The reader is referred there for definitions of separable group (s.g.), coseparable group (c.g.), and weakly separable group (w.s.g.). The first section of this paper establishes general theorems relating the structure of a topological group G to properties of a normal subgroup H and corresponding factor group G/H. For example, if H and G/H are locally bicompact, or if they satisfy the second countability axiom, or if they are complete in the sense of A. Weil, then G has that property. If G is Abelian, and H and G/H are completely zero-dimensional, so is G.

In § 2 it is quickly established that separable, coseparable and weakly separable groups are complete in the sense of Weil, and are completely zero-dimensional, i.e., possess arbitrarily small group-neighborhoods. It is proved that a continuous homomorphic mapping of a w.s.g. upon a w.s.g. is an open mapping. The term "correct subgroup" is introduced to denote a separable subgroup of a w.s.g. whose associated factor group is a c.g. and it is shown that the sum and intersection of two such subgroups is also correct. This and other results of the section are used to show that a w.s.g. is the union of a countable set of subgroups which satisfy the second countability axiom; finally, there are some theorems on topological spaces which have an analogous property. In § 3 it is proved that the character group of a separable group is coseparable, of a coseparable group separable, and of a weakly separable group weakly separable. The key theorem is demonstrated that if G is w.s. with character group X, then G is isomorphic to the character group of X.

The remainder of the paper is devoted to primary groups associated with some prime p. The following theorems occur in § 4. Theorem 12 asserts that if G is primary and w.s. and if H is a discrete subgroup such that $pH=H$, then H is a direct factor of G; theorem 12* is to similar effect with H open. Theorem 14 is: if G is w.s. and H is a subgroup such that G/H is coseparable and the direct sum of groups of type J_p, then H is a direct factor. Theorem 15: every c.g. G contains an open subgroup H such that H is a direct sum of groups of type J_p and finite cyclic groups, and G/H is a direct sum of group of type p^∞. Theorem 16: every w.s.g. G contains a c.g. H such that G/H is locally separable. The section closes with some structure theorems on so-called reduced groups, containing no subgroup T for which $pT=T$. Two "universal" groups are constructed in § 5: one of these contains each w.s.g. as closed subgroup, the other as factor group.

It is shown that if G is separable and $G=p^\infty G$, then the decomposition of G into a direct sum of groups of type p and R_p is unique to within an isomorphism. Next, an extension of earlier results on locally compact groups gives: every regularly stratified, primary, weakly separable G, which is not a group of rank one, decomposes into a direct sum of proper subgroups. In the concluding § 7 it is shown by simple examples that (1) if a group is not locally compact, it may be zero-dimensional without being "completely" so, (2) a complete topological group may be zero-dimensional and not have arbitrarily small subgroups, and (3) if a complete topological group does not satisfy the second countability axiom, it may have factor groups which are not complete. *L. Zippin* (Flushing, N. Y.).

Vilenkin, N. Ya. Corrections to the paper: "Direct decompositions of topological groups. I." Mat. Sbornik N.S. **22(64)**, 191–192 (1948). (Russian)

The paper appeared in Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 85–154 (1946); these Rev. **8**, 132.

Mackey, George W. The Laplace transform for locally compact Abelian groups. Proc. Nat. Acad. Sci. U. S. A. **34**, 156–162 (1948).

Sur un groupe abélien localement compact G, les caractères généralisent les exponentielles complexes $t\to e^{ixt}$ sur la droite R. L'auteur commence par généraliser de même les exponentielles réelles $t\to e^{xt}$: ce sont les fonctions $t\to e^{x(t)}$, où $t\to x(t)$ est une représentation continue de G dans R: la transposée d'une telle représentation est une représentation continue de R dans le dual $\hat{G}$ de G (sous-groupe à un paramètre de $\hat{G}$); cette notion n'a d'intérêt que lorsque $\hat{G}$ est connexe, cette condition étant nécessaire et suffisante pour que pour tout $t\neq 0$ dans G, il existe une représentation continue x de G dans R telle que $x(t)\neq 0$. Il est alors immédiat de définir la généralisation de la transformation de Laplace pour les fonctions numériques définies sur G (il s'agit naturellement de la généralisation de la transformation $L(f)=\int_{-\infty}^{+\infty} f(t)e^{xt}dt$, où l'intégrale est étendue à toute la droite): c'est une fonction définie dans une partie de $\hat{G}\times\hat{G}$ où $\hat{G}$ est le groupe des représentations continues de G dans R. L'auteur esquisse les définitions de la dérivée d'une application de $\hat{G}$ dans R le long d'un sous-groupe à un paramètre, puis de la notion de fonction analytique définie dans $\hat{G}\times\hat{G}$ et donne une condition pour que la transformée de Laplace d'une fonction donnée soit analytique en ce sens. Il indique enfin dans quelle direction on peut chercher des généralisations des formules d'inversion de la transformation de Laplace classique. Aucune démonstration n'est donnée. *J. Dieudonné* (Nancy).

NUMBER THEORY

Jarden, Dov. Inequalities in Fibonacci's sequence. Riveon Lematematika **2**, 35 (1948). (Hebrew)

Five inequalities for the product $u_a u_b$ of two terms of the Fibonacci sequence $1, 1, 2, 3, 5, \cdots$ are given. They are derived from the identities $u_a u_b = u_{a+b-1} - u_{a-1}u_{b-1} = u_{a+b-2} + u_{a-2}u_{b-2}$. *D. H. Lehmer* (Berkeley, Calif.).

Whitlock, W. P., Jr. Pythagorean variations. Scripta Math. **12**, 259–265 (1946).

Bell, E. T. A Pythagorean variation. Scripta Math. **13**, 163–168 (1947).

In the second paper Bell gives a complete solution of the Diophantine equation $(x^2-w)^2+(y^2-w)^2=(z^2-w)^2$ explic-

itly in terms of nine independent integral parameters. An infinitude of solutions is given by Whitlock in the first paper.
I. Niven (Eugene, Ore.).

Kesava Menon, P. On the equation $ax^2 \pm by^2 = cz^2$. Math. Student **14** (1946), 77–80 (1948).

By means of hyperbolic sines and cosines, from an integral solution of $ax^2 - by^2 = 1$ infinitely many solutions are derived. Using circular functions, from a solution of $ax^2 + by^2 = (cz^2)^q$ for $q = 1$ there are derived solutions for $q > 1$. Special cases.
N. G. W. H. Beeger (Amsterdam).

Kesava Menon, P. Some Diophantine equations. Math. Student **14** (1946), 65–68 (1948).

Elementary theorems on systems $\sum_{i=1}^{n} A_i x_i^r = 0$, $r = 0, 1, \cdots, m-1$. For example, if $x_i = a_i$ $(i = 1, \cdots, n)$ is a solution, then

$$x_i = (m+1)a_i \sum_{j=1}^{n} A_j a_j^m - \sum_{j=1}^{n} A_j a_j^{m+1}, \quad i = 1, \cdots, n,$$

is a solution of the system for $r = 0, 1, \cdots, m-1, m+1$.
N. G. W. H. Beeger (Amsterdam).

Palamà, Giuseppe. Multigrade fattoriali. Revista Unión Mat. Argentina **13**, 3–11 (1948).

If $\sum_{i=1}^{p} a_i^k = \sum_{i=1}^{p} c_i^k$, $k = 1, \cdots, n$, then for any $\alpha_1, \cdots, \alpha_n$:

$$\sum_{i=1}^{p} (a_i + \alpha_1)(a_i + \alpha_2) \cdots (a_i + \alpha_k) = \sum_{i=1}^{p} (c_i + \alpha_1) \cdots (c_i + \alpha_k), \quad k = 1, \cdots, n.$$

The author calls such a system a "factorial multigrade equality" and gives some theorems.
N. G. W. H. Beeger (Amsterdam).

Moessner, Alfred. Einige Diophantische Probleme und Resultat. Bull. Calcutta Math. Soc. **39**, 73–78 (1947).

Solutions of some equations $x^n + y^n = (x-y)^{n+1}$, $x^3 + y^3 = (x+y)^2$, $\sum_{i=1}^{4} a_i^n = \sum_{i=1}^{4} b_i^n$, $n = 1, 2, 4$. Numerical examples.
N. G. W. H. Beeger (Amsterdam).

Pompeiu, D. Les sommes de nombres entiers. Acad. Roum. Bull. Sect. Sci. **23**, 225–228 (1942).

Elementary proof of the impossibility of $(x-a)^3 + x^3 = (x+a)^3$ in integers. *N. G. W. H. Beeger* (Amsterdam).

Nasir, Abdur Rahman. Properties of three numbers in A-G.P. Bull. Calcutta Math. Soc. **39**, 79–81 (1947).

Five elementary results are obtained which concern numbers of the sequence A, $(A+D)R$, $(A+2D)R^2$, $(A+3D)R^3$, $\cdots$. For example, if a, b, c, d are any four consecutive members of such a sequence then $(bc-ad)^2 = 4(b^2-ac)(c^2-bd)$.
D. H. Lehmer (Berkeley, Calif.).

Jacobsthal, Ernst. Correction de quelques erreurs dans la table d'indices de M. Kraitchik. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 1, 1–2 (1946).

Eleven errors are noted in Kraitchik's table of indices of primes congruent to 100 modulo p, $p \leqq 10^4$ [Recherches sur la Théorie des Nombres, v. 1, Gauthier-Villars, Paris, 1924, pp. 216–264], for $p = 8191$ and 9137. *D. H. Lehmer.*

Storchi, Edoardo. Un teorema dedotto dal teorema di Wilson che caratterizza i numeri primi della forma $12h+7$. Boll. Un. Mat. Ital. (3) **2**, 229–231 (1947).

The author deduces from Wilson's theorem and some elementary congruence relations the result that a necessary and sufficient condition that a number of the form $12h+7$ be prime is that

$$(3 \cdot 5 \cdot 7 \cdots (4h+1))^2 (2h+1)! \equiv (-1)^{\mu+1} \bmod (12h+7),$$

where μ is the number of quadratic nonresidues less than $(12h+7)/2$. *R. Bellman* (Stanford University, Calif.).

Rankin, R. A. The difference between consecutive prime numbers. III. J. London Math. Soc. **22** (1947), 226–230 (1948).

Let p_n denote the nth prime and let

$$l = \liminf_{n \to \infty} (p_{n+1} - p_n)/\log p_n.$$

It follows from the prime number theorem that $l \leqq 1$, and the author has shown [part II, Proc. Cambridge Philos. Soc. **36**, 255–266 (1940); these Rev. **1**, 292] that if θ is the lower bound of all positive numbers σ_0 such that no Dirichlet L-function $L(s, \chi)$ has a zero at $s = \sigma + it$, where $\sigma > \sigma_0$, then $l \leqq (1+4\theta)/5$. Erdös [Duke Math. J. **6**, 438–441 (1940); these Rev. **1**, 292] established the existence of a number $c < 1$ such that $l \leqq c < 1$. He made use of Brun's method. In this note the author points out that the improved results of Buchstab [Rec. Math. [Mat. Sbornik] N.S. (4) **46**, 375–387 (1938); C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 544–548 (1940); these Rev. **2**, 348] may be used to show that $l \leqq 57/59$. *R. D. James* (Vancouver, B. C.).

Linteş, Ion. Sur la distribution des nombres premiers. Acad. Roum. Bull. Sect. Sci. **26**, 83–88 (1946).

This contains a number of assertions concerning the distribution of the primes, with "proofs" which are entirely fallacious. *H. Davenport* (London).

Erdös, P., and Turán, P. On some new questions on the distribution of prime numbers. Bull. Amer. Math. Soc. **54**, 371–378 (1948).

For $a, b > 0$ let $M_t(a, b) = ((a^t + b^t)/2)^{1/t}$ $(t \neq 0)$,

$$M_0(a, b) = \lim_{t \to 0} M_t(a, b) = (ab)^{\frac{1}{2}}.$$

Let p_n be the nth prime. It is proved that, for fixed real t, each of the inequalities $M_t(p_{n-1}, p_{n+1}) > p_n$, $M_t(p_{n-1}, p_{n+1}) < p_n$ has an infinity of solutions. In particular $(t = 0)$, $\log p_n$ is neither a convex nor a concave function of n for all sufficiently large n. The proof is elementary. A transcendental method, based on the distribution of primes in arithmetical progressions, is also indicated, and it is suggested that this might possibly be used to attack some unsolved problems which naturally arise. *A. E. Ingham.*

Turán, Paul. On certain exponential sums. Nederl. Akad. Wetensch., Proc. **51**, 343–352 = Indagationes Math. **10**, 132–141 (1948).

The main result is as follows. If $-\frac{1}{2} \leqq \gamma \leqq 1$ $(\gamma \neq 0)$, $c \leqq t^{10} \leqq \frac{1}{2}N \leqq N' < N'' \leqq N$, $t > 0$, then

$$(*) \qquad \left| \sum_{N' \leq p \leq N''} e^{it \log^{\gamma+7} p} \right| < C\left(\frac{N \log^8 N}{(|\gamma| t)^{\frac{1}{2}}} + \frac{N \log^8 N}{|\gamma| t} \right),$$

where p runs through primes, and c, C are numerical constants. The author has discussed elsewhere the relation of the sum on the left, with $\gamma = 0$, to the "quasi-Riemannian hypothesis" (the existence of a $\vartheta < 1$ such that $\zeta(s)$ has at most a finite number of zeros in $\sigma > \vartheta$), and has shown how an estimate of this sum, with $\gamma \neq 0$, may be obtained by Vinogradov's method without reference to the zeta-function [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 197–262 (1947); these Rev. **9**, 80]. This estimate

is now obtained (in the slightly improved form (*)) from the theory of $\zeta(s)$ and the known facts about the distribution of its zeros, with the help of ideas introduced by Hoheisel in his study of $\pi(x+h)-\pi(x)$. [Cf. Čudakov, Doklady Akad. Nauk SSSR (N.S.) **58**, 1291–1294 (1947); these Rev. **9**, 333, and the reviewer's note on the possibility of simplification by the use of integrals instead of sums.]

A. E. Ingham (Cambridge, England).

Rodosskiĭ, K. A. On the distribution of prime numbers in short arithmetic progressions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 123–128 (1948). (Russian)

This paper is concerned with the determination of the number of primes in comparatively short arithmetic progressions. More precisely, assuming $(D, l)=1$, and letting $\pi(x, D, l)=\sum 1$ summed over $p\leqq x$, $p\equiv l \pmod D$, the author proves the following theorem. Let ϵ be any small fixed positive number and $\log^{1+\epsilon} D\leqq\log x\leqq\log^{9/4} D$, $D\geqq D_0(\epsilon)$. Then

$$\pi(x, D, l)=\frac{1}{\varphi(D)}\sum_{2\leq n\leq x}1/\log n-E\frac{\chi_1(l)}{\varphi(D)}\sum_{2\leq n\leq x}n^{\beta_1-1}/\log n+\frac{x}{\varphi(D)\log x}O\left(\exp\left\{-A(\epsilon)\frac{\log x}{\log D}\right\}\right),$$

where $A>0$ depends on ϵ; the O is uniform in D; χ_1 is a certain exceptional character; $E=0$ or 1 and β_1 is a zero of $L(s, \chi_1)$. An analogous result is given for $\psi(x, D, l)=\sum\Lambda(n)$ summed over $n\leqq x$, $n\equiv l \pmod D$, where $(D, l)=1$, $\Lambda(n)=\log n$ for $n=p^a$ and 0 otherwise. In order to obtain the theorem stated above the author utilizes his previous results [same Izvestiya Ser. Mat. **12**, 47–56 (1948); these Rev. **9**, 413] concerning upper estimates for the number of L-series (with characters modulo D) which have zeros in certain rectangles.

H. N. Shapiro (New York, N. Y.).

Čudakov, N. G. On the limits of variation of the function $\psi(x, k, l)$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 31–46 (1948). (Russian)

It has been proved by Ingham [Quart. J. Math., Oxford Ser. **8**, 255–266 (1937)] that

$$\pi(x+u)-\pi(x)\sim u(\log x)^{-1},\qquad u=x^\theta,\ \theta>(1+4c)/(2+4c),$$

if $\zeta(\frac{1}{2}+ti)=O(|t|^c)$. Titchmarsh [Quart. J. Math., Oxford Ser. **13**, 11–17 (1942); these Rev. **4**, 131] proved this result for all $c>19/116$, thus confirming Ingham's theorem for all $\theta>48/77$. Using Ingham's method the author extends these results to primes in an arithmetic progression. He obtains in the usual notation

$$\pi(x+u; k, l)-\pi(x; k, l)\sim u(\varphi(k)\log x)^{-1}$$

provided that $u=x^\theta$, $\log k=O(\log x)^{1-\tau}$, where $\tau>\frac{3}{4}$ is an arbitrary constant. This holds uniformly in k (for fixed k the generalization of Ingham's theorem would be almost trivial). Starting from the inequality $L(\frac{1}{2}+ti)=O(k|t|^c)$ the author proves that the Dirichlet L-series $L(s, \chi)$ has at most $O(k^{2(1-\sigma_1)}T^{b(1-\sigma_1)}\log^b(kT))$ zeros in the rectangle $\sigma_1\leqq\sigma<1$, $|t|\leqq T$, where χ is a character mod k and $b>2(1+2c)$ is a constant. From this inequality the result follows in the classical way, but a further complication arises from the fact that for given k one $L(s, \chi)$ may have a real zero near $s=1$, which requires separate treatment. The paper suffers from various misprints and minor inaccuracies.

H. Heilbronn (Bristol).

Roth, K. F. A theorem involving squarefree numbers. J. London Math. Soc. **22** (1947), 231–237 (1948).

The author proves that every large integer n may be represented as the sum of a squarefree number and the square of a squarefree number. He derives the following asymptotic formula for the number $G(n)$ of such representations:

$$G(n)=6\pi^{-2}n^{\frac{1}{2}}\left\{\prod_{p^2\mid n}\left(1-\frac{1}{p+1}\right)\right\}\left\{\prod{}'\left(1-\frac{v(n, p^2)}{p^2-1}\right)\right\}+O(n^{\frac{1}{4}+\epsilon}),$$

where $\prod'$ is over p which do not divide n and

$$v(n, p^2)=\begin{cases}1+(n|p), & p>2,\\ 1+(-1)^{(n-1)/2}, & p=2,\end{cases}$$

and where $(n|p)$ denotes the Legendre symbol. A similar theorem, giving an estimation of the number $H(n)$ of squarefree numbers not exceeding n that are of the form z^2+k, where k is fixed and z squarefree, is stated. The method is elementary and was suggested by some earlier work of Estermann on the representations of an integer as the sum of a square and a squarefree number [Math. Ann. **105**, 653–662 (1931)].

A. L. Whiteman (Los Angeles, Calif.).

Ostmann, Hans-Heinrich. Untersuchungen über den Summenbegriff in der additiven Zahlentheorie. Math. Ann. **120**, 165–196 (1948).

If $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ denote sets of nonnegative integers, the Schnirelmann sum $\sum\mathfrak{A}_\nu$ consists of all the numbers of the form $\sum a_\nu$ with $a_\nu\in\mathfrak{A}_\nu$. The set $\mathfrak{A}_0$ is called a relative zero of $\mathfrak{A}$ if $\mathfrak{A}=\mathfrak{A}+\mathfrak{A}_0$. The set $\mathfrak{A}$ has relative zeros other than $\{0\}$ if and only if $\mathfrak{A}=\mathfrak{B}_0+\mathfrak{B}_1$, where $\mathfrak{B}_0=\{0, a, 2a, \cdots\}$ and $\mathfrak{B}_1$ is a subset of a complete residue system (mod a); $a>0$. Such a set $\mathfrak{A}$ has a largest relative zero $\mathfrak{C}_0$ which can be described in a similar fashion. The relative zeros of $\mathfrak{A}$ are identical with those subsets of $\mathfrak{C}_0$ that contain zero.

A set $\mathfrak{A}$ is called primitive if $\mathfrak{A}=\mathfrak{B}+\mathfrak{C}$ implies that at least one of the sets $\mathfrak{B}$ and $\mathfrak{C}$ contains less than two elements. Various sufficient conditions for primitivity are given. The following one seems typical. Let $\mathfrak{A}=\{a_0, a_1, a_2, \cdots\}$; $0\leqq a_0<a_1<a_2<\cdots$. Suppose there is an index k so that $a_{\lambda+1}$ exists and that $a_{\lambda+1}-a_\lambda>a_k-a_0>0$ for all $\lambda\geqq k$. Then $\mathfrak{A}$ is primitive. The author proves that every set can be decomposed into a (finite or infinite) sum of primitive sets. Neither the decomposition nor the number of terms need be unique.

P. Scherk (Saskatoon, Sask.).

Bellman, Richard. On the average value of arithmetic functions. Proc. Nat. Acad. Sci. U. S. A. **34**, 149–152 (1948).

The author announces a new method of estimating sums of the form $S=\sum_{n=1}^N f(g(n))$, where $f(n)$ is an arithmetic function such as the Euler φ-function, and $g(n)$ is either a polynomial in n, a term of the form ap_n+b, where p_n is the nth prime, or a term of the form ak^n+b. The point of the method is to express $f(n)$ in terms of Ramanujan's function $c_q(n)$ and then, for the polynomial case, to use Hua's result on exponential sums [J. Chinese Math. Soc. **2**, 301–312 (1940); these Rev. **2**, 347]. An asymptotic formula for $\sum_{n=1}^N\sigma(g(n))$ is proved, where $\sigma(n)$ is the sum of the divisors of n and $g(n)$ is a polynomial in n of degree k with integral coefficients having no factor in common. [In the theorem, n is misprinted x, and in lemma 1, $\sum a_k x^k$ should be $\sum a_l n^l$.]

R. D. James (Vancouver, B. C.).

Bellman, Richard, and Shapiro, Harold N. The algebraic independence of arithmetic functions. I. Multiplicative functions. Duke Math. J. 15, 229–235 (1948).

A multiplicative arithmetical function $f(n)$ is one defined for all positive integers n and such that $f(mn)=f(m)f(n)$ whenever m and n are relatively prime. The problem considered is that of excluding, under general conditions, the existence of an algebraic relation $P(f_1(n), \cdots, f_N(n))=0$ for all n, where $P(x_1, \cdots, x_N)$ is a polynomial with real coefficients, not all zero. The main theorem proved is that, if such a relation holds, then every sequence n_i for which $(n_i, n_j)=1$ for $i\neq j$ contains a subsequence such that $f_1^{a_1}(n)\cdots f_N^{a_N}(n)=f_1^{b_1}(n)\cdots f_N^{b_N}(n)$ for all n of the subsequence. The two sets of exponents here consist of non-negative integers and are not the same. The theorem is applied to prove that the functions n, $\phi(n)$, $\sigma(n)$, $d(n)$, $2^{\nu(n)}$ are algebraically independent, and it is also proved that $\mu(n)$ may be added to this set. *H. Davenport* (London).

Wade, L. I. Algebraic independence of certain arithmetic functions. Duke Math. J. 15, 237 (1948).

The author gives a short and direct proof that the arithmetical functions n, $\phi(n)$, $\sigma(n)$, $d(n)$, $2^{\nu(n)}$ are algebraically independent [see the preceding review]. He uses the lemma that $f_1(n), \cdots, f_k(n)$ are certainly algebraically independent if there exists a sequence n_i of positive integers such that $f_j(n_i)\to\infty$ and, for every λ, $f_j(n_i)/f_{j+1}^{\lambda}(n_i)\to\infty$ as $n_i\to\infty$. *H. Davenport* (London).

Titchmarsh, E. C. On series involving divisors. J. London Math. Soc. 22 (1947), 179–184 (1948).

In this note the author investigates the asymptotic behavior of the function $f(z)=\sum d(n)z^n=\sum z^n/(1-z^n)$, as z approaches the unit circle, by a method which avoids the use of the Riemann zeta-function. The substitution $z=\exp(2\pi ih/k-\delta)$, where $\Re(\delta)>0$, changes $f(z)$ into $F(\delta, h, k)=\sum_{r=1}^{k}\sum_{m=0}^{\infty}g_r(m)$, where $g_r(m)=z^{r+mk}/(1-z^{r+mk})$. Using the summation formula

$$\lim_{N\to\infty}\left[\sum_{m=0}^{N}g_r(m)-\int_0^N g_r(x)dx\right]=\tfrac{1}{2}g_r(0)+\int_0^{\infty}(x-[x]-\tfrac{1}{2})g_r'(x)dx,$$

the sum for $F(\delta, h, k)$ splits into four parts, three of which are estimated directly with appropriate error terms. The fourth is in integral form and is evaluated by suitably altering the path of integration. Different paths of integration are required for different ranges of $\arg\delta$. A sample result is $F(\delta, h, k)=1/(k\delta)(\log(1/\delta)-2\log h+\gamma)+O(k\log k)$ for $k^2|\delta|\leqq\pi$, $0\leqq\arg\delta\leqq\frac{1}{4}\pi$. *R. D. James.*

Amante, Salvatore. Sulle funzioni analitiche numerico-integrali di una o più funzioni numeriche. Boll. Un. Mat. Ital. (3) 2, 109–117 (1947).

This is a contribution to a calculus of number-theoretical functions originated by M. Cipolla [Atti Accad. Gioenia Catania (5) 8, no. 11 (1915)] in which

$$(f\times g)(n)=\sum_{d|n}f(n/d)g(d),$$

α is the function equal to 1 for $n=1$ and 0 for $n>1$, and the "powers" $f^{\times n}$ are defined recursively by $f^{\times n+1}=f^{\times n}\times f$, $f^{\times 0}=\alpha$. The author [Rend. Circ. Mat. Palermo 60, 373–378 (1936)] proved that, if $\sum a_kz^k$ has radius of convergence r, the series $\sum_k a_kf^{\times k}(n)$ is absolutely convergent if and only if $|f(1)|<r$. Here he applies this result, first to generalize an analogue of the Taylor series given by Cipolla, then to define and discuss the "powers" $f^{\times s}(n)$ for arbitrary real s and finally to define the "trigonometric functions" obtained by replacing z^k by $f^{\times k}(n)$ in the usual power series. The author then extends his convergence theorem and "Taylor series" to functions of two variables, that is, to series involving two functions $f(n)$, $g(n)$. *R. P. Boas, Jr.*

Amante, Salvatore. Sulle serie numerico-integrali. Matematiche, Catania 1, 217–219 (1946).

The author proves the necessity of a sufficient condition which he gave [cf. the preceding review] for the convergence of Cipolla's "numerical" analogue of a Taylor or Laurent series. *R. P. Boas, Jr.* (Providence, R. I.).

Chowla, S. On series of the Lambert type which assume irrational values for rational values of the argument. Proc. Nat. Inst. Sci. India 13, 171–173 (1947).

The author shows that the function

$$g(x)=x/(1-x)-x^3/(1-x^3)+x^5/(1-x^5)-\cdots$$

has an irrational value when x has the form $1/t$, where $t\geqq 5$ is a positive integer. To this end he considers the sum $S=\sum_{n=1}^{\infty}r(n)/t^n$, where $r(n)$ is the number of representations of n as a sum of two squares. Using the "decimal" scale with basis t he shows by means of well-known arithmetical theorems that S is irrational for an integer $t\geqq 5$. The result stated above then follows on account of the relation $4g(x)=\sum_{n=1}^{\infty}r(n)x^n$. *J. Popken* (Utrecht).

Varnavides, P. Note on non-homogeneous quadratic forms. Quart. J. Math., Oxford Ser. 19, 54–58 (1948).

Let $f(x, y)=ax^2+bxy+cy^2$ be an indefinite binary quadratic form with real coefficients and discriminant $d=b^2-4ac>0$. Let $\varphi(f)$ denote the lower bound of the numbers M for which the inequality $|f(x, y)|\leqq M$ always has a solution in real x, y satisfying $x\equiv x_0 \pmod 1$, $y\equiv y_0 \pmod 1$, where x_0, y_0 are arbitrarily given real numbers. Now the author improves some results of Davenport, who gave a method for estimating an upper bound for $\varphi(f)$ in general cases [Nederl. Akad. Wetensch., Proc. 49, 815–821 = Indagationes Math. 8, 518–524 (1946); these Rev. 8, 444] and he applies this improvement in a particular case which had been considered by Heinhold [Math. Z. 44, 659–688 (1939)]. Improvements of Heinhold's results are given. *J. F. Koksma.*

Hlawka, Edmund. Über Potenzsummen von Linearformen. Akad. Wiss. Wien, S.-B. IIa. 154, 50–58 (1945).

Let $L_1, \cdots, L_n$ be linear forms in $x_1, \cdots, x_n$ of determinant $D\neq 0$, consisting of r forms with real coefficients and s pairs of forms with conjugate complex coefficients $(r+2s=n)$. Let $q\geqq 2$ be real. Minkowski, in his Geometrie der Zahlen, gave an inequality of the form $|L_1|^q+\cdots+|L_n|^q\leqq A|D|^{q/n}$, which always has a solution in integers $x_1, \cdots, x_n$, not all zero. Blichfeldt [Math. Ann. 101, 605–608 (1929)] gave two improvements on this when $q=2$. The first of these was extended to $q>2$ by van der Corput and Schaake [Acta Arith. 2, 152–160 (1936)]. The author now extends the second and more precise result of Blichfeldt. His conclusion is that Minkowski's A can be replaced by $\frac{1}{2}(1+n/q)^{q/n}A/(1+r)^{q/n}$, where

$$r=\frac{q}{q+2n}2^{-n/q}-K^{n+1}\left\{\frac{n}{q}K^{1-q}+\frac{n+q}{q}K^{-1-q}-\frac{4n(n+q)}{q(q+2n)}\sqrt{2}\right\},$$

and $K=2^{1-1/q}-1$. *H. Davenport* (London).

Hlawka, Edmund. Über einen Satz aus der Geometrie der Zahlen. Akad. Wiss. Wien, S.-B. IIa. **155**, 75–82 (1947).

The theorem in question is of an "alternative" type, and generalizes results of Mordell and of the author [Math. Z. **49**, 285–312 (1942); these Rev. **5**, 201]. Let M_1, M_2 be bounded, closed sets of points in n-dimensional space with Jordan contents V_1, V_2. Let $k_0, k_1, \cdots, k_n$ be positive integers and suppose that $V_1+V_2>k_0k_1\cdots k_n$. Let M_1' be the set of points $((x_1-y_1)/k_1, \cdots, (x_n-y_n)/k_n)$, where $x=(x_1, \cdots, x_n)$ and $y=(y_1, \cdots, y_n)$ are points of M_1; and similarly for M_2'. Let M_3 be the set of all points given by the same formula where x now belongs to M_1 and y to M_2 or vice versa. Suppose there are at most k_0-r_i-1 points, other than 0, with integral coordinates in M_i', where $i=1$ or 2, and where $r_i\geqq 0$ and $k_0-r_i>V_i/k_1\cdots k_n$. Then the theorem asserts that the set M_3, after all integral translations, provides a covering of space in which every point is covered at least $r+1$ times, where $r=[\frac{1}{2}(r_1V_1+r_2V_2)(V_1V_2)^{-1}]$. The proof is elementary and arithmetical. *H. Davenport.*

Mahler, K. On lattice points in polar reciprocal convex domains. Nederl. Akad. Wetensch., Proc. **51**, 482–485 = Indagationes Math. **10**, 176–179 (1948).

Let k be a plane convex domain, symmetric with respect to the origin, and K the polar reciprocal of k with respect to the circle $x^2+y^2=1$. Let $\Delta(k)$ and $\Delta(K)$ denote the lower bounds of the determinants of the k-admissible and K-admissible lattices, respectively. The author shows by elementary methods that $\frac{1}{2}\leqq\Delta(k)\Delta(K)\leqq\frac{3}{4}$. By computation of the constants for special domains the inequalities are shown to be the best possible. *D. Derry.*

Tietze, Heinrich. Über die Herstellung einer Basis für die ganzen Zahlen eines algebraischen Zahlkörpers. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 147–162 (1944).

Let R be the rational field, g a rational integer and p a prime. If $g^{p-1}\not\equiv 1 \pmod{p^2}$, then the numbers $\alpha_\nu=(g^\nu/q_\nu)^{1/p}$, where q_ν is the maximum pth power of a rational integer which divides g^ν and $\nu=0, \cdots, p-1$, form an integral basis for $R(g^{1/p})$. If $g^{p-1}\equiv 1 \pmod{p^2}$, it suffices to replace one of the α's by a tentatively determined rational linear combination of them with denominator p. This follows by a simple computation from U. Wegner's determination [J. Reine Angew. Math. **168**, 176–192 (1932)] of the discriminants of all fields $R(g^{1/p})$. Dedekind [J. Reine Angew. Math. **121**, 40–123 (1900)] solved the case $p=3$.

G. Whaples (Bloomington, Ind.).

ANALYSIS

Ahiezer, N. I. The constructive theory of functions in the Harkov University and Mathematical Institute (1917–1947). Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 158–174 (1947). (Russian)

This survey deals with problems of best approximation, developments in series of orthogonal functions, the moment problem, almost periodic functions and generalized Fourier integrals.

Beckenbach, E. F. Convex functions. Bull. Amer. Math. Soc. **54**, 439–460 (1948).

This address gives an account, with an extensive bibliography, of the theory of convex functions and its connections with inequalities, the behavior of the mean values of the modulus of an analytic function, Schwarz's lemma, subharmonic functions, differential geometry and other topics. Some new results are announced. *R. P. Boas, Jr.*

Fuchs, Ladislas. A new proof of an inequality of Hardy-Littlewood-Pólya. Mat. Tidsskr. B. **1947**, 53–54 (1947).

The author establishes the following generalization of a known result, using a simple new method of proof. Let $f(x)$ be a convex function of x in an interval I, and let $a_1, \cdots, a_n$; $b_1, \cdots, b_n$ be values in I satisfying $a_1\geqq\cdots\geqq a_n$, $b_1\geqq\cdots\geqq b_n$. If, for fixed real numbers $\lambda_1, \cdots, \lambda_n$, we have $\sum_{k=1}^r\lambda_ka_k\leqq\sum_{k=1}^r\lambda_kb_k$, $1\leqq r\leqq n$, the sign of equality holding for $r=n$, then we have $\sum_{k=1}^n\lambda_kf(a_k)\leqq\sum_{k=1}^n\lambda_kf(b_k)$. In the previously known result, the λ_k were restricted to have value 1. *E. F. Beckenbach* (Los Angeles, Calif.).

Mikusiński, Jan G.-. Sur les moyennes de la forme $\psi^{-1}[\sum q\psi(x)]$. Studia Math. **10**, 90–96 (1948).

Let $\psi(x)$ and $\chi(x)$ be continuous and strictly increasing in an interval I, and let $M\equiv\psi^{-1}[q_1\psi(x_1)+\cdots+q_n\psi(x_n)]$, where the x_j are in I and the q_j are positive and satisfy $q_1+\cdots+q_n=1$. It is well known that we have $M_\psi\leqq M_\chi$ for all (x) and (q) if and only if $\chi\psi^{-1}$ is convex; that is, if and only if χ is convex with respect to ψ. The author presents geometrical, difference and differential interpretations of the above result. Thus it is shown that we have $M_\psi\leqq M_\chi$ for all (x) and (q) if and only if we have $\Delta^2\psi/\Delta\psi\leqq\Delta^2\chi/\Delta\chi$ for all distinct x_1, x_2 in I, where $\Delta\psi\equiv\psi(x_2)-\psi(x_1)$. *E. F. Beckenbach.*

Aczél, J. On mean values. Bull. Amer. Math. Soc. **54**, 392–400 (1948).

A function $M_n(x_1, \cdots, x_n)$ is said to be a mean value function if M_n is continuous, strictly increasing, reflexive, and symmetric. By a theorem of Kolmogoroff and Nagumo, a necessary and sufficient condition that, to a given infinite sequence $M_1(x_1)=x_1, \cdots, M_n(x_1, \cdots, x_n), \cdots$, of mean value functions, there correspond a continuous and strictly increasing function $f(x)$ such that for all n we have (1) $M_n(x_1, \cdots, x_n)=f^{-1}[\{f(x_1)+\cdots+f(x_n)\}/n]$, is that the M_n be connected by an associative property:

$$M_n(x_1, \cdots, x_n)=M_n(M_k, \cdots, M_k, x_{k+1}, \cdots, x_n), \quad k\leqq n,\ n=1, 2, \cdots,$$

where $M_k=M_k(x_1, \cdots, x_k)$. For a fixed n, conditions that a mean value function $M_n(x_1, \cdots, x_n)$ should be of the form (1) were given by Aumann [Math. Ann. **111**, 713–730 (1935)] under the restriction that M_n be analytic. The author now gives a necessary and sufficient condition, without the assumption of analyticity, that a mean value function $M_n(x_1, \cdots, x_n)$ be of the form (1) for a fixed n; this is the bisymmetric condition that the value of the function $M[M(x_{11}, \cdots, x_{1n}), \cdots, M(x_{n1}, \cdots, x_{nn})]$ be unaltered by the interchange of x_{ik} and x_{kl}.

E. F. Beckenbach (Los Angeles, Calif.).

Fenyö, Stephan. The inversion of an algorithm. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 25, 91–94 (1947).

G. Aumann [Math. Ann. **109**, 235–253 (1933)] has shown that if $M_n(x_1, \cdots, x_n)$ is a mean value function, then there exists a mean value function $M_{n+1}(x_1, \cdots, x_{n+1})$ which is the

unique solution of the functional equation

$$M_{n+1}(x_1, \cdots, x_{n+1}) = M_{n+1}[M_n(x_2, \cdots, x_{n+1}), M_n(x_1, x_3, \cdots, x_{n+1}), \cdots, M_n(x_1, \cdots, x_n)].$$

The author now constructs a mean value function of $n-1$ variables from which we regain the original M_n on applying the above algorithm of Aumann. It is shown that $z=M_{n-1}(x_1, \cdots, x_{n-1})$ is the unique solution of

$$M_n(x_1, \cdots, x_{n-1}, z)=z.$$

E. F. Beckenbach (Los Angeles, Calif.).

Jecklin, Heinrich. Der Begriff des mathematischen Mittelwertes und die Mittelwertformeln. Vierteljschr. Naturforsch. Ges. Zürich **93**, 35–41 (1948).

A criticism of the different definitions of mean values. The author has the opinion that the most reasonable one is that of Cauchy who regarded as a mean any value lying between the least and the greatest of the given quantities. Chisini's definition [Period. Mat. (4) **9**, 106–116 (1929)]: "Given a function $f(x_1, \cdots, x_n)$, m is a mean of $(x_1, \cdots, x_n)$ if $f(m, \cdots, m)=f(x_1, \cdots, x_n)$" is not satisfactory in this form as it does not guarantee that m lies between the least and greatest of the x's. Counterexample:

$$f(x_1, x_2)=(x_1^3+x_2^3)/x_1x_2.$$

B. de Sz. Nagy (Szeged).

Jecklin, Heinrich. Über mathematische Mittelwerte. Elemente der Math. **3**, 12–17 (1948).

The author discusses some known facts on means, in particular those he has obtained recently in joint work with M. Eisenring [Mitt. Verein. Schweiz. Versich.-Math. **47**, 123–165 (1947); these Rev. **9**, 81]. *B. de Sz. Nagy.*

Jecklin, Heinrich. Zur Systematik der statistischen Mittelwerte. Z. Schweiz. Statistik u. Volkswirtschaft **83**, 340–347 (1947).

Cf. the preceding review.

Loewner, Charles. A topological characterization of a class of integral operators. Ann. of Math. (2) **49**, 316–332 (1948).

Let $x(t)$, $y(t)$ be continuous functions with period 2π. The closed oriented curve $x=x(t)$, $y=y(t)$ in the (x, y)-plane is said to be generated by the kernel $k(t)\epsilon L(0, 2\pi)$, if $y(t)=-\int_0^{2\pi}k(u)x(t-u)du$. Various problems in analysis and geometry (study of smoothing operators, of the behaviour of analytic functions on the boundary and of umbilical points) lead to the following question: which kernels $k(t)$ generate only curves of nonnegative circulation, i.e., curves γ whose order relative to any point not on γ is nonnegative?

The principal result of the present paper is to show that the following condition is necessary and sufficient: $k(t)$ is, after possible modification on a set of zero measure, analytic in the open interval $(0, 2\pi)$ and its derivative can be represented there by a Laplace-Stieltjes integral $k'(t)=\int_{-\infty}^{\infty}e^{-rt}d\mu(r)$ with a nondecreasing $\mu(r)$. The sufficiency is proved first for a piecewise constant $\mu(r)$ with only a finite number of jumps. One then passes to the general case by an obvious limiting process. It turns out that the degenerate case is equivalent to the following theorem. Let $A(r)=a_0r^n+a_1r^{n-1}+\cdots+a_n$ and $B(r)=b_0r^{n-1}+b_1r^{n-2}+\cdots+b_{n-1}$ be two polynomials such that $a_0>0$, $b_0>0$, the zeros of $A(r)$ and $B(r)$ are all simple and real and are mutually separated. Then all curves $x=A(d/dt)f(t)$, $y=B(d/dt)f(t)$, where $f(t)$ is an arbitrary n times continuously differentiable function of period 2π, are of nonnegative circulation. For the special case $A(r)=r^2-1$, $B(r)=r$, this was proved by G. Bol [Math. Z. **49**, 389–410 (1944); these Rev. **7**, 29]. The proof of the necessity is made by reduction to some known moment problems.

B. de Sz. Nagy (Szeged).

Mandelbrojt, Szolem. Quelques théorèmes de composition. C. R. Acad. Sci. Paris **226**, 1155–1157 (1948).

A theorem on asymptotic series is used to prove the following theorem 1. The positive even integers are divided into two classes $\{\lambda_n\}$, $\{\mu_n\}$, $\mu_1=0$, $m_n>0$, $m_n'>0$ $(n\geqq 0)$, $\{m_n^c\}$, $\{m_n'^c\}$ regularised sequences with respect to logarithms (i.e., $\{\log m_n^c\}$ is the largest convex sequence $\{\alpha_n\}$ with $\alpha_n\leqq\log m_n$), $\sum(m_n^cm_n'^c/m_{n+1}^cm_{n+1}'^c)=\infty$. Then either the even function $f(x)$ vanishes identically, if for $-\infty<x<\infty$ (1) $\int_0^x f(t)dt=O(1)$, $|f^{(n)}(x)|<m_n$ $(n\geqq 0)$, $f^{(\lambda_n)}(0)=0$ $(n\geqq 1)$, or the even function $g(x)$ must vanish identically if $|g^{(n)}(x)|<m_n'$ $(n\geqq 0, -\infty<x<\infty)$, $g^{(\mu_n)}(0)=0$ $(n\geqq 1)$. Theorem 2: $\{\lambda_n\}$ a sequence of positive even integers, $m_n>0$, $\log m_n$ a convex function of n, $\sum m_n/(m_{n+1}\lambda_{n+1})=\infty$; then an even function $f(x)$ satisfying (1) is identically zero. Theorem 3. Hypotheses: $|f^{(n)}(x)|\leqq M_n(n\geqq 0, -\infty<x<\infty)$; $\{\nu_k\}$ is a sequence of even integers, $\{\mu_k\}$ a sequence of odd integers, $f^{(\nu_k)}(0)=0$, $f^{(\mu_k)}(0)=0$; $\psi_1(r)=\exp\{2\sum_{\nu_k<r}\nu_k^{-1}\}$, $\psi_2(r)=\exp\{2\sum_{\mu_k<r}\mu_k^{-1}\}$; a, b, A, B are positive constants, $\alpha_1(r)$, $\alpha_2(r)$ nonincreasing functions such that $\psi_1(r)\geqq A(r/\alpha_1(r))^a$, $\psi_2(r)\geqq B(r/\alpha_2(r))^b$; $\{M_n^c\}$ is the regularisation of $\{M_n\}$ with respect to logarithms. Conclusion: if $\sum\{\alpha_1(n)(M_n^c)^{1/an}\}^{-1}=\sum\{\alpha_2(n)(M_n^c)^{1/bn}\}=\infty$, then $f(x)\equiv 0$. The basic ideas of the proofs are derived from recent papers of the author and N. Wiener and from a paper by the reviewer [Mandelbrojt and Wiener, same C. R. **225**, 978–980 (1947); **226**, 47–49 (1948); Fuchs, J. London Math. Soc. **22**, 19–25 (1947); these Rev. **9**, 230, 229].

W. H. J. Fuchs (Ithaca, N. Y.).

Pastides, Nicolas. Sur la régularisation des fonctions. C. R. Acad. Sci. Paris **226**, 620–621 (1948).

This is a contribution to S. Mandelbrojt's theory of regularization of functions [cf. La Régularisation des Fonctions, Hermann, Paris, 1938]. For every regularizing function $\omega(t)$ the author studies the class C_ω of functions $y=f(x)$ having the properties: (1) $f(x)$ is defined and finite for $x\geqq 0$, (2) to every x_1 there corresponds at least one t_1 with $\omega(t_1)\geqq x_1$ and $f(x)\geqq f(x_1)+t_1(x-x_1)$ for $x<\omega(t_1)$ and (3) $f(x)=f(x+0)$ (the existence of $f(x+0)$ follows from the previous properties). Several properties of this class of functions are stated, e.g., (I) every function of C_ω is its own regularized with respect to $\omega(t)$; (II) if $F(x)$ is the ω-regularized of $\psi(x)$ then $\phi(x)=F(x+0)$ is in C_ω.

A. Dvoretzky (Princeton, N. J.).

Calculus

***Goodstein, R. L. A Text-Book of Mathematical Analysis. The Uniform Calculus and its Applications.** Oxford, at the Clarendon Press, 1948. xii+475 pp. $9.00.

By considering uniform continuity from the very outset instead of continuity, and uniform differentiability instead of a continuous derivative, the author is enabled to deal rigorously with as wide a class of functions as are contem-

plated in the usual intuitive course. Many other admirable procedures are devised to give a rigorous and yet not too difficult treatment. Unfortunately, however, by a needless reordering of topics and by complete avoidance of intuitive notions even for purposes of motivation the author weakens his book very much indeed. This is especially true of chapter X, Differential geometry of a plane curve. The fundamental theorem of the calculus is postponed to near the end of the book and follows rather than precedes applications.
N. Levinson (Copenhagen).

Climescu, Al. C. Notes d'analyse. I. Sur la définition du logarithme dans le domaine réel. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] **2**, 81–88 (1947).
Nous nous proposons de déduire toutes les propriétés importantes de $\log x$ en partant de $\log x = \lim n(x^{1/n}-1)$.
Extract from the paper.

Mahajani, G. S. Generalisation of the expansions of $\log(1+x)$, $(1+x)^m$ and e^x. J. Univ. Bombay (N.S.) **16**, part 3, sect. A, 1–10 (1947).
In this paper the author obtains power series expansions for $\log(1+a_1x+a_2x^2+\cdots)$, $(1+a_1x+a_2x^2+\cdots)^n$ and $\exp(1+a_1x+a_2x^2+\cdots)$, where the coefficients of the expansions are expressed as determinants. The expansion for $\log(1+a_1x+a_2x^2+\cdots)$ was given without proof in a previous note [Mahajani and Behari, same J. (N.S.) **15**, part 5, sect. A, 1–2 (1947); these Rev. **9**, 177]. Formulas for the coefficients of $(1+a_1x+a_2x^2+\cdots)^n$ are given by I. J. Schwatt [Introduction to the Operations with Series, Philadelphia, 1924, p. 123]. Schwatt's formulas, however, are not explicitly in the determinant form of the present paper.
T. Fort (Athens, Ga.).

Bowman, F. Note on the integral $\int_0^{\frac{1}{2}\pi}(\log\sin\theta)^n d\theta$. J. London Math. Soc. **22** (1947), 172–173 (1948).
The author evaluates a more general integral, in which $\sin\theta$ in the integrand is replaced by $2\sin\theta/a$, in terms of an n-rowed determinant involving the sums $\sigma_r=\sum_{m=1}^{\infty}(-)^{m-1}/m^r$. The integral of the title is obtained when $a=2$, but it is worth noting that for $a=1$ the corresponding integrals are expressed in much more compact form.
M. C. Gray.

Bunickij, Eugen. Sur une formule du calcul intégral. Časopis Pěst. Mat. Fys. **72**, 129–130 (1947). (French. Czech summary)
The author observes that

$$F_n(x)=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}x^{n-1-r}\int t^r\varphi(t)dt$$

is a solution of $f^{(n)}(x)=\varphi(x)$.
R. P. Boas, Jr.

Rajagopal, C. T. On the remainder in Taylor's theorem. Math. Student **14** (1946), 71–73 (1948).
In this paper the author obtains a determinant relationship between the remainders of three functions when individually expanded by Taylor's formula. By particularizing two of the functions there result a variety of forms for the remainder in Taylor's formula, including commonly known ones. In a second theorem the author gives determinant inequalities connecting the remainders in case the necessary derivatives do not exist to assure the validity of his first equation. All proofs are elementary.
T. Fort.

Selmer, Ernst S. An approximate formula for $\Gamma(z)$. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 7, 22–24 (1947).

Theory of Sets, Theory of Functions of Real Variables

Duffin, R. J. Nonlinear networks. IIb. Bull. Amer. Math. Soc. **54**, 119–127 (1948).
[For part IIa cf. the same Bull. **53**, 963–971 (1947); these Rev. **9**, 285.] The author considers an n-dimensional transformation $y_i=s_i(x_1,\cdots,x_n)$, $i=1,\cdots,n$. The transformation is written in the form

$$y_1=p_1(x_1, x-x_2, x_1-x_3,\cdots, x_1-x_n),$$

$$y_2=p_2(x_2-x_1, x_2, x_2-x_3,\cdots, x_2-x_n),\cdots,$$

$$y_n=p_n(x_n-x_1, x_n-x_2,\cdots, x_n).$$

The set of functions $p_i(t_1,\cdots,t_n)$, $i=1,\cdots,n$, are considered for all values of the variables. The notation p_{ij} is used when p_i is considered as a function of t_j only, the other variables being held fixed. The author proves that if (a) p_i is continuous; (b) p_{ij} is either an increasing function unbounded at $\pm\infty$, or is constant; (c) there is a sequence of integers $a, b,\cdots, g, h$ (dependent on i) such that each function $p_{ia}, p_{ab}, p_{bc},\cdots, p_{gh}, p_{hh}$ is unbounded at $\pm\infty$, then for any given point y_i there is a unique solution x_i. Other results are given.
N. Levinson (Copenhagen).

Corominas, Ernest. Sur un théorème de M. Denjoy. C. R. Acad. Sci. Paris **226**, 1159–1161 (1948).
Denjoy's theorem [J. Math. Pures Appl. (7) **1**, 105–240 (1915)] can be stated as follows. The four extreme derivates of a function $f(x)$ on $[a,b]$ have these properties almost everywhere: (1) if two derivates on the same side are finite, then all four derivates are equal; (2) two derivates on the same side are not infinite with the same sign; (3) if a derivate is finite, then it is equal to the opposite derivate; (4) if a derivate is infinite, then the opposite derivate is infinite with the opposite sign. Denjoy [Fund. Math. **25**, 273–326 (1935)] defined the "nth differential coefficient" $f_n(x)$ of a continuous function $f(x)$ on $[a,b]$. If at x_0 the $f_k(x_0)$ for $k=1,\cdots,n-1$ exist, the four "nth extreme differential coefficients" of $f(x)$ at x_0 are defined, for instance

$$D_n^+f(x_0)=\limsup_{h\to 0+}\left\{f(x_0+h)-f(x_0)-\sum_{k=1}^{n-1}\frac{h^k}{k!}f_k(x_0)\right\}\frac{n!}{h^n},$$

and, if these four coefficients coincide, they give $f_n(x_0)$. Denjoy [loc. cit., pp. 293–303] proved that for the nth extreme differential coefficients (1) still holds and he asked whether the other above properties also hold. Roger [C. R. Acad. Sci. Paris **214**, 942–944 (1942); these Rev. **5**, 114 (not mentioned by the author)] has already shown that (2) is not valid for nth extreme differential coefficients ($n>1$). The author now proves (sketching only the main idea of the proof) that (2), (3), (4) cannot be generalized for nth extreme differential coefficients ($n>1$).
A. Rosenthal (Lafayette, Ind.).

Zagorskiĭ, Z. On Jordan curves possessing a tangent at every point. Mat. Sbornik N.S. **22**(64), 3–26 (1948). (Russian)
[The author's name appears as Zahorski in non-Russian publications.] Continuing work of Ward [Fund. Math. **28**, 280–288 (1937)] the author gives conditions for the existence of certain kinds of parametrizations of a Jordan curve C in Euclidean n-space E_n. In the following statements when $P=f(t)$ is a parametrization of C, $f'(t)$, when it exists, is the derivative, $\lim_{h\to 0}[f(t+h)-f(t)]/h$; B_t is the set of t

where $f'(t)=0$ (i.e., where all coordinates of $f'(t)=0$) and $f(B_t)$ is the set of all $f(t)$ where t is in B_t. (1) Each continuous rectifiable curve $P=F(T)$ has a parametric representation $P=f(t)$ everywhere differentiable with $f'(t)$ bounded and with the Lebesgue measure of B_t and the Carathéodory linear measure of $f(B_t)$ both zero. [Ward had shown that $f'(t)$ exists almost everywhere.] Corollary: a parametrization of C, $P=f(t)$, with the properties of the theorem exists if and only if there is a parametrization $P=F(T)$ with all the coordinate functions of $F(T)$ of bounded variation; t can then be given as a continuous, nondecreasing function of T. (2) The curve C has a parametrization $P=f(t)$ with $f'(t)$ existing everywhere and the measure of B_t equal to zero if and only if C has a parametrization $P=F(T)$ where the coordinate functions of $F(T)$ all belong to the class VBG^* [as defined by Saks, Theory of the Integral, Warsaw, 1937, p. 228]; t can then be given as a continuous increasing function of T. Other corollaries and one theorem on curves of a special structure are also given. *M. M. Day.*

Orlicz, W. Sur les fonctions satisfaisant à une condition de Lipschitz généralisée. I. Studia Math. **10**, 21–39 (1948).

Let $\omega(h)$, $\omega_1(h)$ decrease on $[0, l]$, vanish only at 0 and approach 0 with h. Let $\gamma(h)=\sup[k/\omega(k)\,|\,0<k\leqq h]$. Let $f(x)$ be real and bounded for all real x. The principal result is as follows. There is an $f(x)$ of period l satisfying (1) $|f(x+h)-f(x)|\leqq\omega(|h|)$, $|h|\leqq l$, and (2) $\limsup_{h\to 0}|f(x+h)-f(x)|/\omega_1(|h|)=+\infty$ for all x if and only if (3) $\liminf_{h\to+0}\omega_1(h)\gamma(h)/h=0$. The sufficiency is established by an effective construction for $f(x)$ and also by means of Baire category. For the latter method there is considered the complete metric space of sequences $q=(q_n)$, where $q_n=0$ or 1 and the distance is $(q', q'')=\sum_1^\infty 2^{-n}|q_n'-q_n''|$. On this basis, if ω, ω_1 satisfy (3); $\limsup_{h\to+0} h/\omega_1(h)<+\infty$; $\phi(x)$ is a nonconstant Lipschitz function of period l; $\alpha_n>0$ and $\beta_n\uparrow+\infty$ are such that $\limsup_n \alpha_n/\omega_1(l\beta_n^{-1})=+\infty$ and $\sum_1^\infty\alpha_n\beta_n\gamma(l\beta_n^{-1})<+\infty$, then $f_q(x)=\sum_1^\infty q_n\alpha_n\phi(\beta_n x)$ has the properties: $|f(x+h)-f(x)|\leqq M\omega(|h|)$ for $|h|\leqq l$ and (2′) $\limsup_{h\to+0}|f(x+h)-f(x)|/\omega_1(h)=+\infty$ for all x and a set $[q]$ of the second category. Among the applications are the following. (i) [Zygmund, Mathesis Polska **4**, 1–7 (1929); Steckel, Mathesis Polska **4**, 123–127 (1929).] There is an $f(x)$ of period l satisfying (1) and $\limsup_{h\to0}|f(x+h)-f(x)|/h=+\infty$ for all x if and only if $\lim_{h\to0} h/\omega(h)=0$. (ii) If $s>1$, then $f(x)=\sum_1^\infty 2^{-n^s}\phi(2^{n^s}x)$ is continuous and nondifferentiable. (iii) There is an $f(x)$ satisfying the Hölder condition: (1) with $\omega(h)=h^\gamma$, $0<\gamma<1$, and (2′) for all x if and only if $\liminf_{h\to+0}\omega_1(h)/h^\gamma=0$. (iv) The function of (ii) satisfies the Hölder condition for $\gamma<1/s$ but, for $\delta>1/s$, $\limsup_{h\to+0}|f(x+h)-f(x)|/h^\delta=+\infty$.

L. W. Cohen (Flushing, N. Y.).

Sibirani, Filippo. Come definire la media aritmetica delle distanze dei punti di un arco da un punto o da una retta? Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **8**, 193–199 (1941).

This note gives many examples which illustrate the proposition that, given a curve, there is no unique, easily defensible choice for "the" arithmetic mean distance from the curve to a point except, perhaps, when the curve is rectifiable.

M. M. Day (Princeton, N. J.).

Hopf, Eberhard. Kennzeichnung der durch Punkttransformationen erzeugten linearen Funktionaloperatoren. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 233–236 (1947).

Let $\mathfrak{M}$ be the set of all bounded real-valued functions on a set X and let L be a one-to-one mapping of $\mathfrak{M}$ onto itself such that L is additive, both L and L^{-1} are nonnegative, and $L1=1$. The author proves that under these hypotheses there exists a one-to-one transformation T of X onto itself such that $Lf(x)=f(Tx)$ for every f in $\mathfrak{M}$. [In this connection see J. von Neumann, Ann. of Math. (2) **33**, 587–642 (1932), in particular, theorem 3, p. 621.]

P. R. Halmos (Chicago, Ill.).

Rohlin, V. A "general" measure-preserving transformation is not mixing. Doklady Akad. Nauk SSSR (N.S.) **60**, 349–351 (1948). (Russian)

If X is the unit interval and μ is Lebesgue measure, then a one-to-one, measure-preserving transformation of X onto itself is called mixing if $\lim_n \mu(T^nA\cap B)=\mu(A)\mu(B)$ for every pair (A, B) of measurable subsets of X; T is called weakly-mixing if

$$\lim_n n^{-1}\sum_{i=1}^{n}|\mu(T^iA\cap B)-\mu(A)\mu(B)|=0.$$

Let M be the set of all mixing transformations and let M^* be the set of all weakly mixing transformations; clearly $M\subset M^*$. The reviewer has shown [cf. Ann. of Math. (2) **45**, 786–792 (1944); these Rev. **6**, 131] that, with respect to a certain natural topology of the set of all measure preserving transformations, the larger class M^* is very large (i.e., is an everywhere dense G_δ); the author's principal result in this paper is that the smaller class M is very small (i.e., is of the first category). *P. R. Halmos* (Chicago, Ill.).

***Hahn, Hans, and Rosenthal, Arthur. Set Functions.** The University of New Mexico Press, Albuquerque, N. M., 1948. ix+324 pp. $12.50.

Despite the fact that real-valued set functions occur in many parts of analysis, until the appearance of this book the only even approximately complete and easily available treatment of them has been that of Saks's Theory of the Integral [Warsaw, 1937]. Since over half of Saks's book is devoted to the study of point functions, it is not surprising that the book of Hahn and Rosenthal goes into much more detail and gives many more illuminating examples of set functions and their behavior. Both the quality and quantity of material in this book are such as to make it a valuable addition to the library of every student of the field.

The book is divided into an introduction (containing an exposition of the relevant parts of set theory and point set topology) and five chapters, entitled (I) Additive and totally additive set functions, (II) Measure, (III) Measurable functions, (IV) Integration and (V) Differentiation. The authors' treatment is thorough, rigorous and exhaustive; it includes, with minor exceptions, the first four chapters of Saks's book and differs from those chapters mainly in the wealth of detail. The set functions studied are defined either in a perfectly abstract set, or else in metric spaces (and, in particular, Euclidean spaces); the intermediate cases of locally compact spaces, normal spaces and topological groups are not discussed.

Very worth-while special features of the book include several excursions into cognate topics (e.g., Hamel bases and convex functions), detailed and up to date references

presented (at the end of each section) so as to be most helpful, and many footnotes beginning with the sentence "This condition is essential" and ending with interesting examples.

A somewhat unfortunate tendency is the use of too much specialized notation and terminology. [It is remarked in the preface that the notation of Hahn's Reelle Funktionen [Akademische Verlagsgesellschaft, Leipzig, 1932] is followed, "even in such cases where other mathematicians have preferred different notations."] As examples of this tendency may be mentioned (1) $A\mathfrak{M}$ (the class of all those subsets of the set A which belong to the class $\mathfrak{M}$), and (2) strongly content-like measure (a measure in a metric space, such that each measurable set may be approximated arbitrarily closely from above by an open set; such a measure is sometimes called regular).

It is disappointing also that the treatment is not well motivated. For instance, the concept of the integrability of a function f with respect to a measure μ is defined (without any preliminary discussion) by the requirement that there exist a countably additive set function λ such that whenever $a \leqq f(x) \leqq b$ on a measurable set E, then $a\mu(E) \leqq \lambda(E) \leqq b\mu(E)$. The only other aspect of the book that might be called a flaw is its somewhat old-fashioned point of view, as exhibited by the terminology, the notation and, to some extent, the choice of topics. *P. R. Halmos* (Chicago, Ill.).

Gomes, Ruy Luís. Some properties of ordinate sets. Gaz. Mat., Lisboa **8**, no. 34, 1–3 (1947). (Portuguese)

The following theorems are established. Let S be the ordinate set of a function f real and bounded on $[a, b]$. The inner measure of S is the least upper bound of measures of closed ordinate sets contained in S. The closure of S is the ordinate set of lim sup f, and S is closed if and only if f is upper semicontinuous. *E. J. McShane.*

Pettineo, B. Sulle funzioni integrabili secondo Picone-Stieltjes. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 230–235 (1947).

If $f(p)$ is a real-valued point function of the space S of one dimension, $\alpha(T)$ an additive interval function of bounded variation on S and Δ an arbitrary subset of S, then $\int_\Delta f d\alpha$ is said to exist in the Picone sense [Teoria Moderna dell'Integrazione delle Funzioni, Pisa, 1945] provided $\lim_C \int_C f d\alpha$ exists, where the class $[C]_{f,\Delta}$ consists of the closed subsets of Δ on which f is continuous, $\int_C f d\alpha$ is a Riemann-Stieltjes integral and the limit is taken in the Moore-Smith sense over the class $[C]_{f,\Delta}$ ordered by inclusion. The note derives properties of this integral similar to the well-known properties of the Lebesgue-Stieltjes integral, such as absolute continuity and complete additivity.
T. H. Hildebrandt (Ann Arbor, Mich.).

Pettineo, B. Sulla sommabilità delle funzioni integrabili secondo Picone-Stieltjes. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 236–241 (1947).

The author expects to consider in a series of notes the relationship between the existence of the Picone-Stieltjes integral $\int_\Delta f(P) d\alpha$ and the existence of $\int_\Delta |f(P)| dV_\alpha$ (for the definition of the integral see the preceding review), V_α being the total variation function of the additive interval function $\alpha(T)$ on the one-dimensional space S. This note concerns itself with the interval function $\alpha(T)$ and its total variation V_α and demonstrates that if C is any closed bounded set on the linear interval then meas $C = \int_C dV_\alpha$ can be expressed as $\lim_{D \to S} \sum_{i=1}^n |\alpha(T_i)|$, where D is a subdivision of S by means of a set of points J of S having no limiting points, T_i are intervals determined by D containing points of C and the limit is a Moore-Smith limit, ordering of subdivisions D being by inclusion. *T. H. Hildebrandt.*

Viola, T. Sui fondamenti geometrici del teorema del cambiamento di variabili negli integrali a più dimensioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 277–282 (1947).

The author states and solves a geometrical problem which arose in establishing the formula for changing variables in an n-dimensional integral, assuming that the transformation involved is biunique, of class C', and has a Jacobian which does not vanish on the range of integration.
R. G. Helsel (Columbus, Ohio).

Simonsen, W. On the transformation of integrals of real functions in abstract space. Mat. Tidsskr. B. **1947**, 55–61 (1947). (Danish)

A careful analysis of the conditions for changes of variable and measure in abstract integrals. *W. Feller.*

Cesari, Lamberto. La nozione di integrale sopra una superficie in forma parametrica. Ann. Scuola Norm. Super. Pisa (2) **13** (1944), 77–117 (1948).

Suppose F is a real valued continuous function on Euclidean 6-space for which $F(x, y, z, ku, kv, kw) = kF(x, y, z, u, v, w)$ whenever $k > 0$. Let S be a map with finite Lebesgue area of the closed square Q into 3-space and let φ_1, φ_2, φ_3 be the maps into the plane obtained by superimposing on S the projections into the coordinate planes. For each simple closed polygon $P \subset Q$ let $\tau_i(P)$ be the integral over the plane of the topological index with respect to the curve obtained by restricting φ_i to P, let $t_i(P) = |\tau_i(P)|$ and $t(P) = (\sum_{i=1}^3 [\tau_i(P)]^2)^{\frac{1}{2}}$. Define $T_i(S)$ and $T(S)$ as the suprema of the sums of t_i and t over all finite sets of simple closed polygons contained in P and with disjoint interiors.

Theorem 1 states that there exists a number $I(S)$ such that $\sum_{j=1}^n F[x_j, y_j, z_j, \tau_1(P_j), \tau_2(P_j), \tau_3(P_j)]$ is close to $I(S)$ whenever $P_1, \cdots, P_n$ are simple closed polygons contained in Q and with disjoint interiors for which the S image of the interior of each P_j has a small diameter, $\sum_{j=1}^n t_i(P_j)$ and $\sum_{j=1}^n t(P_j)$ are close to $T_i(S)$ and $T(S)$, the 2-dimensional measure of $\varphi_i(\cup_{j=1}^n P_j)$ is small, and (x_j, y_j, z_j) is the S image of some point interior to P_j. Theorem 2. If S_1 and S_2 are Fréchet equivalent, then $I(S_1) = I(S_2)$. Theorem 3. If S_1, S_2, S_3, $\cdots$ converge uniformly and in Lebesgue area to S, then $I(S_n) \to I(S)$ as $n \to \infty$, and the convergence of the approximating sums to $I(S_n)$, in the sense of theorem 1, is uniform. Theorem 4. If φ_1, φ_2, φ_3 are absolutely continuous, then $I(S)$ is the integral over Q of the function obtained from F by substituting the three components of S and the generalized Jacobians of φ_1, φ_2, φ_3. *H. Federer.*

***Radó, Tibor. Length and Area.** American Mathematical Society Colloquium Publications, vol. 30. American Mathematical Society, New York, 1948. v+572 pp. $6.75.

This book gives a systematic and detailed exposition of most of the contributions to the theory of the Lebesgue area which have been made by C. B. Morrey, T. Radó and Radó's students. Part I deals with background material in analytic topology and in the theory of Lebesgue

integration, giving statements without proofs. Kuratowski's characterization of the 2-sphere is perhaps the least familiar of these statements.

Part II is devoted to topology. It begins with a discussion of upper semicontinuous collections, monotone-light factorizations and their relation to Fréchet equivalence, the cyclic monotone images of 2-cells and 2-spheres and the approximation of monotone maps by homeomorphisms. Next the theory of cyclic elements of Peano spaces is developed on the basis of conjugacy, including the cyclic chain approximation theorem and the cyclic decomposition of mappings. This leads to a cyclic additivity theorem for certain functionals on the space of all maps of a fixed Peano space into a fixed metric space. A theory of the topological index of maps into the plane is then developed in terms of angular variation. Here and elsewhere the use of combinatorial topology is avoided. Detailed proofs are provided in this and the succeeding parts. Also, each part contains a section of expository discussion.

Part III begins with a theory of interval functions, their Burkill integrals, their derivatives and their completely additive extensions. This is followed by a discussion of total variation and absolute continuity, both in the classical sense and in the sense of Tonelli, the Banach-Vitali theorem on the multiplicity integral, basic theorems on arc length, Steiner's inequality, some results on convergence in length and the formula expressing the integral over the plane of the topological index with respect to a curve as a line integral over the curve. Part IV deals with maps of plane Jordan regions into the plane, essential multiplicity, essential maximal model continua, bounded variation and absolute continuity of such maps, generalized Jacobians, and the transformation of plane integrals.

Part V is devoted to the area of surfaces represented by maps of a 2-cell or a 2-sphere into 3-space. It deals first with the area $\mathfrak{a}$ defined as follows. Consider the function relating to each region the square root of the sum of the squares of the essential multiplicity integrals of those three maps of the region into the plane which are obtained by superimposing the projections into the coordinate planes on the map representing the surface. The value of $\mathfrak{a}$ at a surface is the supremum of all sums of the function just considered over all sets of disjoint subregions of the domain of the map representing the surface. The question raised in the book whether the essential multiplicity integrals may be replaced by the measures of the essential images without altering the supremum may be answered affirmatively, so that $\mathfrak{a}$ coincides with the area of Geöcze. After developing fundamental properties of $\mathfrak{a}$, including integral formulae and the relation to cyclic decompositions, the author establishes the corresponding theorems for the Lebesgue area $\mathfrak{A}$ and also proves the existence of generalized conformal representations for light mappings of finite Lebesgue area. These results are combined to obtain the major theorem which states that $\mathfrak{a}(S)=\mathfrak{A}(S)$ in case $\mathfrak{A}(S)<\infty$ or $\mathfrak{a}(S)=0$. It is pointed out in the book that, in a series of papers only recently accessible in this country, L. Cesari has proved independently that $\mathfrak{a}(S)=\mathfrak{A}(S)$ always. This treatment of general surfaces is followed by a special study of the nonparametric case which includes the basic theory of Tonelli, Radó and Saks, the work of L. C. Young on the upper and lower limits of the integrals obtained by replacing the partial derivatives in the classical integrand by difference quotients, and related results on the problem of Geöcze concerning the approximation in area by inscribed polyhedra. *H. Federer.*

Theory of Functions of Complex Variables

Huber, Heinz. Ein Mittelwertsatz für Funktionen einer komplexen Veränderlichen. Comment. Math. Helv. **21**, 58–66 (1948).

The author establishes the following analogue for analytic functions of the mean-value theorem of the differential calculus. Let $f(z)$ be analytic in $\mathfrak{S}$: $|z-z_0|<S$ and, furthermore, let $f'(z)$ map $\mathfrak{S}$ univalently on a convex domain. Then to every z in $\mathfrak{S}$ there corresponds a unique ξ in $\mathfrak{S}$ for which $f'(\xi)=(f(z)-f(z_0))/(z-z_0)$. Moreover, $\xi(z)$ is analytic and is situated simultaneously in (1) the lens formed by the two circles passing through z and z_0 and touching $|z-z_0|=S$, and (2) the circle of radius $|z-z_0|^2/(2S)$ about $(z+z_0)/2$. [The author seems to be unacquainted with some of the literature on the subject, e.g., P. Montel, J. Math. Pures Appl. (9) **16**, 219–231 (1937), which contains explicitly the result from which the author deduces (1) as a corollary.] *A. Dvoretzky* (Princeton, N. J.).

Bräzma, N. Über eine Riemannsche Fläche. Univ. Riga. Wiss. Abh. Kl. Math. Abt. **1**, 1–21 (1943). (German. Latvian summary)

The Riemann surface of a function w of form $w=(w_1(z)w_2(z))^{\frac{1}{2}}$, where $w_1(z)$ and $w_2(z)$ are each of form $Az^{-1}+Bz+z\sum_1^n A_i(z^2+\omega_i^2)^{-1}$, $i=1, 2$; A, B, A_i, ω_i real, is studied in detail. A single-valued branch of w is defined on one sheet (with cuts on the imaginary axis of z), satisfying the requirements of applications to electric circuit theory, where the w_i appear as impedances [cf. W. Cauer, Math. Z. **38**, 1–44 (1933); H. Piloty, Elektr. Nachr. Techn. **14**, 88–117 (1937)]. *W. Kaplan* (Ann Arbor, Mich.).

Popovăţ, P. Sur une propriété générale de certaines fonctions algébriques. Acad. Roum. Bull. Sect. Sci. **24**, 71–78 (1943).

The author establishes analytical conditions (using Montel's regions of monovalence) for an algebraic correspondence $f(x, y)=0$ between the complex x and y planes to be of the form $A(y)D(x)-B(y)C(x)=0$ where A, B, C, D are polynomials. *O. F. G. Schilling* (Chicago, Ill.).

Biernacki, Mieczysław. Sur une proposition de Bieberbach-Eilenberg. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **1**, 9–12 (1946). (French. Polish summary)

Let $\varphi(w)$ effect a topological transform of the Riemann w-sphere on itself, and let $\varphi(\infty)=a_0$. If F is the class of all functions $f(z)=a_0+a_1z+\cdots+a_nz^n+\cdots$, regular in $|z|<1$ and such that f takes at most one of any pair of values w and $\varphi(w)$, then $|a_1|$ is uniformly bounded in F. The extremal functions are schlicht and map $|z|<1$ on a finite domain D, the frontier of which is transformed in itself by $\varphi(w)$, while its complement is the transform of D by φ. This is a generalisation of a result by Bieberbach [Math. Ann. **77**, 153–172 (1916)] and Eilenberg [Fund. Math. **25**, 267–272 (1935)], where $\varphi(w)=w^{-1}$. The proof, apart from the properties of D, is easy, the class F being obviously normal. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Cremer, Hubert. Dreikreisesatz und Zentrumproblem. Comment. Math. Helv. **21**, 185–188 (1948).

The following theorem is demonstrated with the aid of the three circles theorem. Let $g(z)=a_1z+\cdots$ denote an entire function vanishing at the origin; let $F(r)$ $(>r)$ denote a monotone increasing function of r satisfying $F(r)>\max_{|z|\leq r}|g|$ for large r; let $F_n(r)$ denote the nth iterate of $F(r)$. Then g is

necessarily linear if (1) there exists $\varphi(z)=c_1z+\cdots$ $(c_1\neq 0)$ analytic in a neighborhood of the origin such that g admits the representation $g=\varphi[a_1\varphi^{-1}(z)]$ in some neighborhood of the origin, (2) $\liminf_{n\to\infty}|a_1^n-1|\,|F_n(r)|^{1/n^2}=0$ holds for large r. *M. Heins* (Providence, R. I.).

Castellaneta, Veneranda. Su talune diseguaglianze generali della teoria delle funzioni. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 513–516 (1947).

L'auteur établit que, M_r et $M_{r+\mu}$ désignant le maximum du module d'une fonction $f(z)$ holomorphe dans les cercles C_r et $C_{r+\mu}$, C_r de centre $z=0$ et rayon r, $C_{r+\mu}$ de centre $2+\mu$ et rayon $r+\mu$, $\mu>r$, on a $M_r\leqq M_{r+\mu}(r+\mu)/\mu$, et, si en outre $f(0)=f'(0)=\cdots=f^{(\nu)}(0)=0$, $M_r/M_{r+\mu}\leqq r^{\nu+1}/\{\mu(r+\mu)^\nu\}$. Ce résultat est plus précis que celui déduit par Minetti d'une formule générale [J. Math. Pures Appl. (9) **16**, 155–178 (1937)]. *G. Valiron* (Paris).

Federici, Adele. Un'osservazione sul teorema di Schottky-Bieberbach-Montel. Boll. Un. Mat. Ital. (3) **2**, 231–234 (1947).

Remarks showing how known bounds for the Schottky theorem may be applied to obtain bounds for the Schottky-Bieberbach-Montel theorem in the special case where the functions considered do not vanish. *M. Heins.*

Kufarev, P. P. On the theory of univalent functions. Doklady Akad. Nauk SSSR (N.S.) **57**, 751–754 (1947). (Russian)

The author gives a list of results too complicated to set down here, for certain special classes of univalent functions, without any indication of proofs or motivation. These results appear somewhat akin to Löwner's parametric representation of a class of univalent functions [Math. Ann. **89**, 103–121 (1923)]. *W. Seidel* (Rochester, N. Y.).

Fourês, Léonce. Sur les domaines d'univalence de certaines fonctions entières. C. R. Acad. Sci. Paris **226**, 1157–1159 (1948).

The author describes a method of decomposition of the complex plane into regions of univalence for a given entire function. The method is designed to overcome certain difficulties noted by Valiron [J. Math. Pures Appl. (9) **19**, 339–358 (1940); these Rev. **2**, 358]. *M. Heins.*

Rajagopal, C. T. A note on periodic integral functions. Duke Math. J. **15**, 11–15 (1948).

S'appuyant sur la proposition classique d'après laquelle une fonction entière $f(z)$ admettant la période λ est la somme d'une série $\sum_{-\infty}^{\infty}A_nu^n$, $u=\exp(2i\pi z/\lambda)$, l'auteur montre que: Si $f(z)$ est d'ordre $\rho>1$, les coefficients a_n, b_n de son développement de Fourier vérifient les conditions $a_n=O[\exp(-n^{\alpha-\epsilon})]$, $b_n=O[\exp(-n^{\alpha-\epsilon})]$, où $\alpha=\rho/(\rho-1)$, pour tout $\epsilon>0$. *G. Valiron* (Paris).

Arpiarian, Noubar. Sur une méthode générale de recherche de fonctions orthogonales dans le domaine complexe. C. R. Acad. Sci. Paris **226**, 1335–1337 (1948).

A purely formal substitution method, using the conformal mapping of finite domains on a circular area, is discussed. It allows one to deduce from any sequence of functions orthonormal on a finite or infinite interval another sequence of functions orthonormal (with weight function) on a given curve or inside of a given domain. *E. Kogbetliantz.*

Položii, G. N. On p-analytic functions of a complex variable. Doklady Akad. Nauk SSSR (N.S.) **58**, 1275–1278 (1947). (Russian)

The author says that the complex-valued function $u+iv=f(z)$ is p-analytic in a domain D if $pu_x=v_y$, $pu_y=-v_x$ in D, where p is a positive function of class C^2. He shows that a p-analytic function is an interior transformation and that, along a closed rectifiable arc C in D, $\int_C udZ+ivdZ=0$, where $Z=X+iY$, $\bar{Z}=-\bar{X}+i\bar{Y}$ and X, Y, $\bar{X}$, $\bar{Y}$ are any solutions of the two systems $pX_x=\bar{Y}_y$, $pX_y=-\bar{Y}_x$; $p\bar{X}_x=Y_y$, $p\bar{X}_y=-Y_x$. The proofs are only sketched. If p is of the form $p=\sigma(x)\tau(y)$, then p-analytic functions coincide with the sigma-monogenic functions investigated by Bers and Gelbart [Trans. Amer. Math. Soc. **56**, 67–93 (1944); Ann. of Math. (2) **48**, 342–357 (1947); these Rev. **6**, 86; **8**, 510]. The theorem on the topological equivalence of p-analytic and analytic functions was announced by Bers and Gelbart under the assumption that p is analytic in x and y [Bull. Amer. Math. Soc. **52**, 64 (1946)]. *A. Gelbart.*

Theory of Series

Čelidze, V. G. On the extension of a theorem of Frobenius to double series. Doklady Akad. Nauk SSSR (N.S.) **60**, 553–554 (1948). (Russian)

Let $\lambda\geqq 1$. A double series $\sum a_{jk}$ is summable A_λ to s if the series in

$$(1)\qquad f(x,y)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{jk}x^jy^k$$

converges absolutely when $|x|<1$, $|y|<1$ and defines a function $f(x,y)$ such that $f(x,y)\to s$ as $(x,y)\to(1,1)$ subject to the restrictions $0<x<1$, $0<y<1$, and $\lambda^{-1}\leqq(1-x)/(1-y)\leqq\lambda$. Let

$$(2)\qquad s_{mn}=\sum_{j=0}^{m}\sum_{k=0}^{n}a_{j,k};\quad \sigma_{m,n}=\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}s_{jk}.$$

I. I. Ogieveckiĭ [same Doklady (N.S.) **58**, 1897–1900 (1947); these Rev. **9**, 278] proved that if σ_{mn} converges to s and

$$(3)\qquad \limsup_{m\to\infty}|s_{mn}|/(m+1)<\infty,\quad \limsup_{n\to\infty}|s_{mn}|/(n+1)<\infty,$$

for each n and m, respectively, then $\sum a_{jk}$ is summable A_λ to s. The author, in a paper unavailable to the reviewer, proved that if σ_{mn} converges to s and

$$(4)\qquad \lim_{m\to\infty}\sigma_{m,n}/(m+1)^2=0,\quad \lim_{n\to\infty}\sigma_{mn}/(n+1)^2=0,$$

then $\sum a_{jk}$ is summable A_λ to s.

This note shows very simply that if $\sigma_{mn}\to s$, then (3) implies (4) and hence that the theorem of Ogieveckiĭ is a corollary of that of Čelidze. An example disproves the converse; the theorem of Čelidze is the stronger of the two. [For other work along these lines, see J. C. Vignaux, An. Soc. Ci. Argentina **126**, 321–344, 401–428 (1938); **127**, 161–185 (1939); **140**, 134–135 (1945), and references given there.] *R. P. Agnew* (Ithaca, N. Y.).

***Turán, P. On power-series whose coefficients form a multiply monotonic sequence.** Semitic Studies in Memory of Immanuel Löw, Budapest, 1947, pp. 300–305.

A sequence $\{c_n\}$ is monotonic of order m if $(-1)^k\Delta^kc_n\geqq 0$ for all $k=0,1,\cdots,m$ and all $n=0,1,2,\cdots$. If $m=1$, then $\{c_n\}$ is nonnegative and decreasing. For a fixed sequence

$\{a_n\}$ and any sequence $\{b_n\}$, let $f(\{b_n\};x)=\sum_0^\infty b_n a_n x^n$. Set $A_n=\sum_0^n a_k$. Then $f(\{b_n\};x)$ is bounded from above (below) on $0\leqq x<1$ for every null sequence $\{b_n\}$ of order 1 if and only if the sequence $\{A_n\}$ is bounded from above (below). This is applied to answer several questions raised by Fejér [Acta Litt. Sci. Univ. Hung. Sect. Sci. Math. **8**, 89–115 (1937)] and S. Sidon, concerning monotonic properties of functions defined by power series with multiply monotonic coefficient sequences. The results can also be stated in terms of the behavior of Abel transforms; in this form, a related problem was treated by Kaczmarz [Studia Math. **3**, 95–100 (1931)]. *R. C. Buck* (Providence, R. I.).

Atkinson, F. V. The Abel summation of certain Dirichlet series. Quart. J. Math., Oxford Ser. **19**, 59–64 (1948).

Using contour integration, the author obtains the following formula: $\sum_1^\infty n^{-s}e^{-n\delta}-\Gamma(1-s)\delta^{s-1}\to\zeta(s)$ as $\delta\downarrow 0$, when s is not a positive integer. He also gives a result of similar type, too long to quote here, relating to the (everywhere divergent) series $\sum p(n)(n-1/24)^{-s}$, where $p(n)$ denotes the number of unrestricted partitions of n. *A. Dvoretzky.*

Upadhyay, S. D., and Das Gupta, P. N. On a generalised continued fraction. Bull. Calcutta Math. Soc. **39**, 65–70 (1947).

Let

$$a_1+\cfrac{1}{a_2+\cdots+\cfrac{1}{a_n+\cdots}}$$

denote a continued fraction in which the partial denominators themselves represent continued fractions, namely,

$$a_1=a_{11}+\cfrac{1}{a_{12}+\cfrac{1}{a_{13}+\cdots}},\quad a_2=a_{21}+\cfrac{1}{a_{22}+\cfrac{1}{a_{23}+\cdots}},\quad \cdots.$$

Let p and q denote the numerators and denominators, respectively, of the convergents

$$a_{11}+\frac{1}{a_{12}}=A\begin{pmatrix}1\\2\end{pmatrix},\quad a_{11}+\frac{1}{a_{21}}=A\begin{pmatrix}1&2\\1&1\end{pmatrix},$$

$$\left(a_{11}+\frac{1}{a_{12}}\right)+\frac{1}{a_{21}}=A\begin{pmatrix}1&2\\2&1\end{pmatrix},$$

etc. The authors write a number of recurrence formulas of the type

$$p\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}=\begin{vmatrix}a_{11}&-1&0\\1&a_{12}&-1\\0&1&a_{13}\end{vmatrix}p\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}+\begin{vmatrix}a_{11}&-1\\1&a_{13}\end{vmatrix}p\begin{pmatrix}1&2\\1&1\end{pmatrix}$$

from the recurrence formulas for the convergents of the continued fraction. They also develop matrix product formulas of the form

$$\begin{bmatrix}p\begin{pmatrix}1&2&\cdots&n\\n&n&\cdots&n\end{pmatrix}&p\begin{pmatrix}1&2&\cdots&n\\n&n&\cdots&n-1\end{pmatrix}\\ q\begin{pmatrix}1&2&\cdots&n\\n&n&\cdots&n\end{pmatrix}&q\begin{pmatrix}1&2&\cdots&n\\n&n&\cdots&n-1\end{pmatrix}\end{bmatrix}=\begin{bmatrix}a_{11}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{12}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{1n}&1\\1&0\end{bmatrix}$$

$$\times\begin{bmatrix}a_{21}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{22}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{2n}&1\\1&0\end{bmatrix}$$

$$\times\cdots\times\begin{bmatrix}a_{n1}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{n2}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{nn}&1\\1&0\end{bmatrix}.$$

The latter are a generalization of the formulas given by L. M. Milne-Thomson [The Calculus of Finite Differences, Macmillan, London, 1933, pp. 109–110]. *E. Frank.*

Thron, W. J. Some properties of continued fractions $1+d_0z+K(z/(1+d_nz))$. Bull. Amer. Math. Soc. **54**, 206–218 (1948).

This deals with continued fractions of the form:

$$(1)\qquad 1+d_0z+\cfrac{z}{1+d_1z+\cfrac{z}{1+d_2z+\cdots}}$$

and related power series: (2) $1+\sum c_nz^n$. It is shown that corresponding to every formal power series (2) there exists exactly one continued fraction of the form (1) whose nth approximant has for each n a power series expansion agreeing with (2) up to and including the term c_nz^n. If the continued fraction (1) converges uniformly in a region D containing the origin, then (2) converges to the same function in the largest open circular region with center at the origin contained in D. Convergence of the power series, however, does not imply convergence of the corresponding continued fraction.

Proofs are then given that various criteria are sufficient or necessary for the convergence or uniform convergence of the continued fraction (1) in regions of various types. Some sample results are: (α) the continued fraction (1) converges uniformly in a region D if and only if the approximants of (1) are ultimately uniformly bounded in every closed subset of D; (β) if $|d_n|\leqq M$, then (1) converges for $|z|\leqq 1/(1+(M+1)^{\frac{1}{2}})^2$, while if $|d_n|\geqq m$, then (1) converges for $|z|\geqq 1/(1-(m+1)^{\frac{1}{2}})^2$, the convergence being uniform on closed subsets of these regions; and (γ) if $\theta\leqq\arg d_n\leqq\phi$, then with certain restrictions on θ and ϕ, (1) converges for z in the angular region $-\pi-2\theta<\arg z<\pi-2\phi$.

O. Frink (State College, Pa.).

Harmonic Functions, Potential Theory

Brelot, Marcel. Deux théorèmes généraux sur le potentiel et quelques applications. C. R. Acad. Sci. Paris **226**, 1499–1500 (1948).

The author states his results with reference to the compact space $\bar{R}_\tau$ of $\tau\geqq 2$ dimensions, obtained from the Euclidean space R_τ by adjoining a point at infinity [see Ann. Sci. École Norm. Sup. (3) **61**, 301–332 (1944); these Rev. **7**, 204]. The following two theorems are stated without proof. (I) Let w be the difference of two functions each subharmonic in Ω, an open domain of $\bar{R}_\tau$. If w is non-negative where defined, then on the set where w is defined and zero, the difference of the masses associated with the subharmonic functions is nonpositive. (II) Let E be a fixed nonpolar point set of the domain Ω of $\bar{R}_\tau$ and let u_0 be subharmonic in the intersection of Ω and an open neighborhood of the boundary of Ω. Consider a set of functions u which are subharmonic in Ω, dominate u_0 and are all dominated on E by the same constant. A necessary and sufficient condition that on each compact set of Ω these functions have a common upper bound is that the absolute value of the total mass associated with u have a fixed upper bound as u ranges over its set. An application is given for each theorem. *F. W. Perkins* (Hanover, N. H.).

Bolder, H. Une extension de la loi de réciprocité pour l'énergie dans la théorie du potentiel et quelques applications. I. Nederl. Akad. Wetensch., Proc. **51**, 496–502 = Indagationes Math. **10**, 180–186 (1948).

If a simple distribution G_2 of charges is assembled in the presence of another fixed distribution G_1 a potential energy

is introduced which the author represents by $E(G_1|G_2)$. The known law of reciprocity states that under certain conditions $E(G_1|G_2)=E(G_2|G_1)$. The author discusses a generalization of this; he replaces G_1 and G_2 by I_1 and I_2, respectively, representing combinations of simple and double distributions. *F. W. Perkins* (Hanover, N. H.).

Deny, Jacques. Un théorème sur les ensembles effilés. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **23**, 139–142 (1948).

The set E in Euclidean n-space R_n is said to have the property $\mathcal{S}$ (effilé) at a point O if there exists a distribution μ of positive mass and if there exists a neighborhood $N(O)$ of O such that $U_\mu(O)<1$ and $U_\mu(M)\geqq 1$, $O\neq M$, $M\in N(O)\cap E$; here U_μ is the Newtonian potential function $U_\mu(P)=\int (PQ)^{-n+2}d\mu(Q)$, $n>2$, and $U_\mu(P)=-\int \log (PQ)d\mu(Q)$, $n=2$. The author proves the following theorem. Let E have the property $\mathcal{S}$ at O, and consider the rays from O whose intersection with E has O as a cluster point. Then these rays meet the unit spherical surface centered at O in a set of zero exterior capacity. The proof depends on a criterion due to Brelot [Bull. Sci. Math. (2) **68**, 12–36 (1944); these Rev. **7**, 15]. The author applies this theorem to the study of pseudo-limits of Newtonian potentials. *M. O. Reade.*

Kampé de Fériet, J. Remarques sur les fonctions orthogonales à toute fonction harmonique dans un domaine plan, à propos des équations du mouvement plan d'un fluide visqueux incompressible. Ann. Soc. Sci. Bruxelles. Sér. I. **62**, 11–18 (1948).

The author has shown that if a function ψ is regular in a suitably restricted region S of the plane and constant on the boundary C of S then its normal derivative is zero on C if and only if its Laplacian is orthogonal to every function harmonic in S [Math. Mag. **21**, 74–79 (1947); these Rev. **9**, 433]. Such functions ψ are of interest in connection with the boundary value problem associated with the two-dimensional flow of an incompressible viscous fluid in a fixed container to the boundary C of which it adheres. The author gives a method for constructing functions of this type, with especial attention to the case in which S is the interior of a unit circle. *F. W. Perkins.*

Protter, M. H. Generalized spherical harmonics. Trans. Amer. Math. Soc. **63**, 314–341 (1948).

The author considers the differential equation

$$(1)\qquad \sum_{i=1}^{3}\{\sigma_i(x_1)\tau_i(x_2)\pi_i(x_3)u_{x_i}\}_{x_i}=0.$$

Following the methods introduced by Bers and Gelbart [Quart. Appl. Math. **1**, 168–188 (1943); same Trans. **56**, 67–93 (1944); these Rev. **5**, 25; **6**, 86] for the equation $\sum_{i=1}^{2}\{\sigma_i(x_1)\tau_i(x_2)u_{x_i}\}_{x_i}=0$, the author obtains the "polynomial" solutions $\bar{R}_{n,\nu}(X^{(k)}, Y^{(l)}, Z^{(m,n)})$ of (1), where $X^{(k)}$, $Y^{(l)}$, $Z^{(m,n)}$ are generalized powers of x, y, z. The following properties of $\bar{R}_{n,\nu}$ are proved: (1) $\bar{R}_{n,\nu}$ is regular in the entire space; (2) $\bar{R}_{n,\nu}$ is obtained by quadratures; (3) every solution $u(x, y, z)$ of (1), regular around the point $P(x, y, z)$ can be expanded uniquely in the form

$$(2)\qquad \sum_{n=0}^{\infty}\sum_{\nu=0}^{2n}A_{n,\nu}\bar{R}_{n,\nu},$$

which converges absolutely in a small enough region, where $A_{n,\nu}$ are constants explicitly given. When (1) reduces to the Laplace equation $u_{xx}+u_{yy}+u_{zz}=0$, the $\bar{R}_{n,\nu}$ become homogeneous harmonic polynomials in x, y, z of degree n and any three-dimensional harmonic function can be expanded by a series of the form (2), where $A_{n,\nu}$ are the Taylor coefficients. The extension of these investigations to higher dimensions introduces no difficulties. *A. Gelbart.*

Differential Equations

***Erougin, N. Reducible systems.** Trav. Inst. Math. Stekloff **13**, 95 pp. (1946). (Russian. English summary)

Consider the differential equation (1) $X'=XP$ whose terms are matrix functions of the time t, and the accent denotes derivative as to t. The unknown is X, and P is given and continuous and bounded in $[t_0, +\infty]$. The system (1) is said to be reducible, according to Liapounoff, whenever there exists a matrix $Z(t)$, bounded together with Z' and the determinant $|Z^{-1}|$ for $t>t_0$, such that the transformation $Y=XZ$ gives rise to an equation (2) $Y'=YB$, where B is constant. The main purpose of the paper is to study a necessary and sufficient condition for reducibility. Consider the adjoint system (3) $T'=-PT$. Theorem 1. A necessary and sufficient condition for reducibility of (1) to (2) is that (3) possess a solution Ze^{-Bt}. Theorem 2. If (1) is reducible to (2) then, upon reducing B to its canonical form, the real part of the canonical form is unique. The reducibility of systems for which P has a limit at infinity is fully dealt with. Generalizations are also given to systems with P unbounded. *S. Lefschetz* (Princeton, N. J.).

Faedo, S. Sulla stabilità delle soluzioni delle equazioni differenziali lineari. IV. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 192–198 (1947).

Continues the first three parts [same Rend. (8) **2**, 564–570, 757–764 (1947); **3**, 37–43 (1947); these Rev. **9**, 285] for the case of multiple roots where the right member is not zero. *N. Levinson* (Copenhagen).

Wallach, Sylvan. The stability of differential equations with periodic coefficients. Proc. Nat. Acad. Sci. U. S. A. **34**, 203–204 (1948).

The author proves that, if $f(t)$ is a real, continuous, non-constant function of period π such that $n^2\leqq f(t)\leqq(n+1)^2$ for some integer n, then the characteristic exponents of $x''+f(t)x=0$ are distinct and of magnitude 1. The author remarks that at least for the case $n^2<f(t)<(n+1)^2$ the theorem is known. *N. Levinson* (Copenhagen).

Wintner, Aurel. Asymptotic integrations of the adiabatic oscillator in its hyperbolic range. Duke Math. J. **15**, 55–67 (1948).

Let $f(t)$ be of bounded variation over $(0, \infty)$ and let $\lim_{t\to\infty} f(t)>0$. Then the differential equation $x''-f(t)x=0$ has a solution $x_1(t)\sim\exp(\int_0^t f(t)^{\frac{1}{2}}dt)$ as $t\to\infty$ and another solution $x_2(t)$ given asymptotically by $\exp(-\int_0^t f(t)^{\frac{1}{2}}dt)$. The first derivatives of $x_1(t)$ and $x_2(t)$ are given asymptotically by differentiating the above relations. Other consequences are given. *N. Levinson* (Copenhagen).

Levinson, Norman. The asymptotic nature of solutions of linear systems of differential equations. Duke Math. J. **15**, 111–126 (1948).

Let A denote an $n\times n$ (complex) constant matrix with n simple roots; $\Phi(t)$, $R(t)$ continuous matrix functions of a real variable t whose elements satisfy $\phi_{ij}(t)\to 0$ as $t\to\infty$,

$\int^{\infty}|d\phi_{ij}|<\infty$ and $\int^{\infty}|r_{ij}|dt<\infty$. Additional restrictions are imposed on the matrix Φ when the real parts of the characteristic roots of A are not distinct. Employing the method of successive approximations, the author deduces an asymptotic formula for solution vectors $x=x(t)$ of the linear system $x'=(A+\Phi+R)x$. The asymptotic formula obtained generalizes a result of Wintner in the case of a second order linear differential equation [Amer. J. Math. 69, 251–272 (1947), pp. 261, 270; these Rev. 9, 35, and the paper reviewed above]. The particular case where $\Phi=0$ above can be read off from a theorem of Dunkel [Proc. Amer. Acad. Arts Sci. 38, 339–370 (1902)] by a suitable change of independent variables. *P. Hartman* (Baltimore, Md.).

Collatz, L. Über Stabilität von Regelungen mit Nachlaufzeit. Z. Angew. Math. Mech. 25/27, 60–63 (1947).

Flügge-Lotz, I., Hodapp, H. F., Klotter, K., Meissinger, H., und Scholz, K. Über Bewegungen eines Schwingers unter dem Einfluss von Schwarz-Weiss-Regelungen. Z. Angew. Math. Mech. 25/27, 97–113 (1947). (German. Russian summary)

In der Technik werden aus Gründen apparativer Einfachheit oft unstetig arbeitende Regelungen gebaut, um Schwinger mit unzureichender Eigendämpfung zu verbessern. Diese Regelungen arbeiten z. B. so, dass bei positivem Ausschlagen des Schwingers eine zusätzliche konstante Rückstellkraft oder ein konstantes Rückstellmoment aufgebracht wird, bei negativem Ausschlagen dasselbe mit entgegengesetztem Vorzeichen. Deshalb gehorcht die Bewegung in zwei aufeinanderfolgenden Intervallen verschiedenen Gesetzen und muss stückweise berechnet werden, ähnlich wie Schwingungen mit Coulombscher Reibung. Das kostet viel Zeit, und es bestand deshalb ein Bedürfnis, Methoden zu entwickeln, die eine Übersicht über den Bewegungsablauf vermitteln, ohne diese stückweise Berechnung von Frequenz und Amplitude zu erfordern.

Authors' summary.

Lehmann, N. Joachim. Die Stabilitätsfrage bei rückgekoppelten Verstärkern. I. Z. Angew. Math. Mech. 28, 23–29 (1948).

Lehmann, N. Joachim. Die Stabilitätsfrage bei rückgekoppelten Verstärkern. II. Z. Angew. Math. Mech. 28, 59–64 (1948).

Im ersten Teil werden die Grundlagen zur rechnerischen und zeichnerischen (experimentellen) Stabilitätsuntersuchung linearer, rückgekoppelter Verstärker von einem einheitlichen mathematisch-physikalischen Standpunkt betrachtet und in eine für die Anwendung geeignete Form gebracht; der zweite Teil bringt dazu Beispiele (Vierpoltheorie) und Ergänzungen. *From the author's summary.*

Četaev, N. G. On the sign of the smallest characteristic number. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 101–102 (1948). (Russian)

In the problem of stability of motion the question of the sign of the smallest characteristic number of the system of linear differential equations representing the motion plays an interesting role. This problem has been considered by A. Liapounoff [Problème Général de la Stabilité du Mouvement, reprinted by Princeton University Press, 1947; these Rev. 9, 34]. In a previous paper [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 323–326 (1944); these Rev. 6, 225] the author considered the case when the coefficients of the differential equations tend to definite limits. The method used can be applied also to special cases when the coefficients are of unbounded variation. The general case for coefficients of unbounded variation was treated by K. P. Persidskiĭ [Rec. Math. [Mat. Sbornik] 40, 284–293 (1933)]. In the present paper the author reduces the problem to an investigation of the characteristic numbers of n expressions involving the coefficients of the differential equations.

Let the motion be represented by the equations

$$dx_s/dt=p_{s1}x_1+\cdots+p_{sn}x_n, \quad s=1, 2, \cdots, n.$$

By the method of successive approximations, the initial values $x_s^{(0)}$ may be written $x_s^{(0)}=\sum_r A_{sr}x_r$, where the A_{sr} are expressed in terms of multiple integrals involving $p_{si}(t_k)$. The matrix $\|c_{\alpha\beta}-\delta_{\alpha\beta}\mu\|$ is then considered. Here $c_{\alpha\beta}=\sum_s A_{s\alpha}A_{s\beta}$, $\delta_{\alpha\beta}$ is the Kronecker δ, and μ is some positive number which exists provided the inequalities $c_r/c_{r-1}>\eta>0$ $(r=1, \cdots, n)$ are satisfied, where $c_0=1$ and c_r is the r-rowed determinant with elements $c_{\alpha\beta}$. The author finds that if all the principal diagonal minors of the above matrix are positive, the undisturbed motion is stable. If the undisturbed motion is asymptotically stable, an unbounded increasing positive function η can be found, and the smallest characteristic number of the given system of differential equations is positive. If η is a vanishing function with a positive characteristic number, the smallest characteristic number of the solutions of the differential equations is negative.

S. D. Zeldin (Cambridge, Mass.).

Nikitin, V. P., Turkin, V. K., and Kunickiĭ, N. P. On the stability of operation of an amplidyne electric drive. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1946, 1567–1580 (1946). (Russian)

The application of the amplidyne in industry and war technology has widely expanded in recent years. However, the mathematical theory of the amplidyne electric drive is very little developed. In this article the authors present a preliminary study concerning the stability and aperiodicity of the amplidyne electric drive and construct diagrams for the study of the transient processes described by the differential equations representing the motion. The problem is reduced to the study of the roots of an algebraic equation of the fourth degree associated with the differential equations.

S. D. Zeldin (Cambridge, Mass.).

Nikitin, V. P., Turkin, V. K., and Kunickiĭ, N. P. Stability diagrams for systems of the fifth order. Doklady Akad. Nauk SSSR (N.S.) 58, 591–594 (1947). (Russian)

A dynamical system whose work is described by differential equations of the form $dw_i/dt=f_i(w_1, \cdots, w_n)$ $(i=1, \cdots, n)$, where $w_1, \cdots, w_n$ are functions of the time t (in electrotechnics they may represent currents, tensions, etc.), is known to have a stationary state if the system of equations $f_i(w_1, \cdots, w_n)=0$ $(i=1, \cdots, n)$ has real roots $w_i=w_i^*$ $(i=1, \cdots, n)$. Otherwise the system is not stationary. The description of the transient process is best illustrated by a stability diagram. In this paper the authors present a method for the construction of such a diagram for $n=5$. In a previous paper this method was used for $n=4$ [cf. the preceding review].

For sufficiently large values of t, the transient process

may be described by the approximate equalities

$$w_k \cong w_k^0 + e^{-\rho t}\sum_{j=1}^{m} c_{kj} \sin(\sigma_j t + \tau_{kj}), \quad k=1, \cdots, n,$$

where ρ is the damping coefficient. A knowledge of ρ helps to determine the rapidity with which the system returns to the stationary state.

In the construction of a stability diagram it is required to indicate graphically the relation between the damping coefficient ρ and the coefficients of an algebraic equation of the nth degree. This is difficult for $n>4$ since the functions involved contain a large number of independent variables. For $n=5$, the algebraic equation reduces, after several appropriate substitutions, to the form

$$w^5+5qw^4+(10q^2+1)w^3+(10q^3+3q+x)w^2 + (5q^4+3q^2+2qx+y)w+q^5+q^3+q^2x+qy+z=0.$$

The authors obtain the Hurwitz conditions for this equation in the form of five inequalities. The diagram for a special case is drawn showing the boundaries of the stability regions to be straight lines and parabolas. *S. D. Zeldin.*

Nikitin, V. P., Turkin, V. K., and Kunickiĭ, N. P. On diagrams exhibiting to what extent the damping of a transient process differs from the damping according to a simple exponential law. Doklady Akad. Nauk SSSR (N.S.) **59**, 1097–1099 (1948). (Russian)

In previous papers [same vol., 57–60 (1948); these Rev. **9**, 352, and the two papers reviewed above] the authors have presented methods for the construction of stability diagrams showing the dependence of the damping coefficient of a transient process on the parameters entering in the differential equations describing the dynamic system. Diagrams were also considered for frequency of oscillations, where the dependence of the circular frequency on the parameters is shown. In the present paper the authors consider the construction of a diagram to show how the damping of the transient process differs from the damping calculated according to the simple exponential law.

Let the dynamic system be represented by the equations $d\omega_i/dt = f_i(\omega_1, \cdots, \omega_n)$ $(i=1, \cdots, n)$; and let the n roots of the algebraic equation $s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n=0$ be the characteristic numbers of the solutions of the equations of the disturbed motion. The algebraic equation is assumed to have no multiple roots. By substitutions $s=u+Q$, $s=q-v$, where Q and q are real $(Q>q)$, the transformed equations are such that if the coefficients of the equation satisfy Hurwitz's conditions of the form $\alpha_i(Q)>0$, $\beta_i(q)<0$ $(i=1, \cdots, n)$, then $q<\Re(s)<Q$. From this it follows that the damping coefficient of the transient process has a value lying between $-q$ and $-Q$. Furthermore, the real parts of any two roots of the algebraic equation differ from each other by not more than ϵ, where $\epsilon=Q-q$. The smaller the value of ϵ, the more regular is the character of the transient process when the system tends to the stationary state. This means that the smaller the value of ϵ, the less is the difference between the character of the transient process and that of the process obtained by the exponential law Ce^{vt}. The diagram which illustrates the dependence of the damping coefficient, as well as the regularity of the transient process, on the parameters in the differential equations is called the diagram of regularity of the transient process. This diagram, when constructed on a coordinate plane, shows the plane broken up into regions bounded by the lines $\alpha_i(Q)=0$ and $\beta_i(q)=0$ $(i=1, \cdots, n)$. *S. D. Zeldin* (Cambridge, Mass.).

Nalli, Pia. L'equazione differenziale

$$y''+y=f(x)\sqrt{1-y^2-y'^2}.$$

Boll. Un. Mat. Ital. (3) **2**, 195–204 (1947).

The author solves the equation given in the title. The analysis is very complicated and the method cannot be briefly sketched. The equation arises in connection with the problem of determining a curve by means of its curvature and torsion, expressed in terms of arc length.

R. Bellman (Stanford University, Calif.).

Levi, Beppo, and Massera, José L. Study in the large of a differential equation of the second order. Math. Notae **7**, 91–155 (1947). (Spanish)

The nonlinear differential equation $yy''=x$ is studied in great detail and essentially all qualitative features of the solutions are obtained. The solutions with the singular initial values $y=y_0\neq 0$, $x=0$ are obtained in power series form; for $y_0=0$, the curve $3y^2-4x^3=0$ is shown to be the unique solution. The differential equation is shown to be invariant under a group of transformations: $x=\lambda^2X$, $y=\lambda^3Y$, and a corresponding reduction to a first order equation is made. The latter equation is then studied thoroughly and the results obtained permit a completion of the analysis of the given second order equation. *W. Kaplan.*

Minorsky, N. Self-excited mechanical oscillations. J. Appl. Phys. **19**, 332–338 (1948).

The equation $\ddot{\theta}+p\dot{\theta}+q\dot{\theta}_h+\omega^2\theta+\mu f(\theta, \dot{\theta}_h)=0$ is considered for small μ, where $\theta=\theta(t)$ and $\theta_h=\theta(t-h)$. Using devices of Kryloff and Bogoliuboff and van der Pol the author obtains approximate results of considerable physical interest.

N. Levinson (Copenhagen).

Minorsky, Nicolas. Sur une classe d'oscillations auto-entretenues. C. R. Acad. Sci. Paris **226**, 1122–1124 (1948).

Special cases of the results of the paper reviewed above.

N. Levinson (Copenhagen).

Graffi, Dario. Sopra alcune equazioni differenziali della radiotecnica. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **9**, 145–153 (1942).

The author considers the equation

$$Li''+f(i)i'+\phi(i)/c=0 \tag{1}$$

which arises in the study of electronic oscillators. The relation

$$L\int_{t_1}^{t_2} i'^2 = \int_{t_1}^{t_2} \{i\phi(i)/c\}dt \tag{2}$$

is derived under the condition $i(t_1)=i(t_2)=0$. From this equality and Wirtinger's inequality a lower bound for the difference between consecutive zeros of i is obtained.

Next the case where i is a periodic solution is considered. From (1) relations between the Fourier coefficients of i and the period are determined. Finally the case where the right-hand side of (1) has a forcing term is discussed. Independently of the author, Shohat has used (2), together with Fourier analysis, to discuss van der Pol's equation. [Cf. J. Appl. Phys. **15**, 568–574 (1944); these Rev. **6**, 1.]

R. Bellman (Stanford University, Calif.).

Plato, G. Über das Verhalten eines angefachten schwingungsfähigen Systems mit einem Freiheitsgrad, dessen Dämpfung dem Quadrat der Geschwindigkeit proportional ist. Z. Angew. Math. Mech. **28**, 91–92 (1948).

The differential equation $\ddot{x}+a\dot{x}^2 \operatorname{sgn} x+bx=\epsilon \sin t$ is treated by approximate methods for small ϵ. It is further assumed that $x(t)$ has exactly two zeros for $0\leqq t<2\pi$.

N. Levinson (Copenhagen).

Baiada, E. Confronto e dipendenza dai parametri degli integrali delle equazioni differenziali. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 258–263, 264–271 (1947).

Every maximal and minimal integral of $y'=f(x, y, \lambda)$, where f is continuous, is respectively upper and lower semi-continuous in the parameter λ and in the initial conditions.

N. Levinson (Copenhagen).

Barbuti, U. Sull'integrale massimo e minimo e sulla unicità della soluzione delle equazioni e dei sistemi differenziali del primo ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 272–276 (1947).

Let $F(x, y)$ be continuous in the rectangle $|x|\leqq a$, $|y|\leqq b$. Let $y=f(x)$ be a solution of $y'=F(x, y)$ in the rectangle such that $f(0)=0$. If, for any continuous $g(x)$ with $g(0)=0$ and $b>g(x)\geqq f(x)$, we have $F(x, g(x))-F(x, f(x))\leqq\{g(x)-f(x)\}/x$ for $|x|\leqq a$, then $y=f(x)$ is the maximal integral through $(0, 0)$. Other results and the generalization to systems are given.

N. Levinson (Copenhagen).

Borg, Göran. Bounded solutions of a system of differential equations. Ark. Mat. Astr. Fys. **34B**, no. 24, 7 pp. (1948).

The author considers the vector differential equation

$$\bar{Y}''+A\bar{Y}=F(t, \bar{Y}, \bar{Y}'),$$

where $A(t)$ is a continuous matrix which is of bounded variation over (t_0, ∞). The quadratic form associated with $A(\infty)$ is positive definite and it is assumed that $\|F\|\leqq f(t)g(\{\|\bar{Y}\|^2+\|\bar{Y}'\|^2\}^{\frac{1}{2}})$, where $g(y)$ is nondecreasing, $\int^\infty\{g(y)\}^{-1}dy=\infty$ and $\int^\infty f(t)dt<\infty$. Then $\bar{Y}$ and $\bar{Y}'$ are bounded as $t\to\infty$. Other results are given.

N. Levinson.

Reid, William T. Addendum to "A matrix differential equation of Riccati type." Amer. J. Math. **70**, 460 (1948).

The paper appeared in the same J. **68**, 237–246 (1946); these Rev. **7**, 446. The author acknowledges that some of his results are contained in papers of J. Radon [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1927**, 243–257; Abh. Math. Sem. Hamburgischen Univ. **6**, 273–299 (1928)].

Ważewski, Tadeusz. Une méthode topologique de l'examen du phénomène asymptotique relativement aux équations différentielles ordinaires. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 210–215 (1947).

The following theorem is established. Let $\dot{x}_i=g_i(t, x_1, \cdots, x_n)$ $(i=1, \cdots, n)$ be a system of differential equations defined in a domain D of the $(n+1)$-dimensional space of the variables; let the g be continuous in D and be such that there is just one solution through each point of D. Let E be a domain contained in D and let S be a subset of $D\cdot(\bar{E}-E)$ such that through each point Q of S there is a solution arc PQ starting at a point P of E and leaving E first at Q as time increases; let the transformation from P to Q be continuous. Let Z be a subset of $E+S$ such that $Z\cdot S$ is not a retract of Z but is a retract of S. Then there is a point P of $Z\cdot E$ such that the solution through P cannot leave E at a point of S as time increases. The author gives two applications of the theorem and states that it can be used to establish and generalize known theorems on asymptotic properties of solutions of differential equations.

W. Kaplan (Ann Arbor, Mich.).

Zwirner, G. Sugli elementi uniti delle trasformazioni funzionali: alcune applicazioni ai problemi ai limiti per le equazioni differenziali ordinarie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 242–247 (1947).

In this note the author states two new and complicated theorems concerning the existence and uniqueness of the solution of a differential equation

$$y^{(n)}(x)=f(x, y(x), y'(x), \cdots, y^{(n-1)}(x))$$

and a set of conditions of the form

$$y(x_1)=y(x_2)=\cdots=y(x_n)=0.$$

One of these theorems amounts to an extension of a theorem given by de la Vallée Poussin [J. Math. Pures Appl. (9) **8**, 125–144 (1929)]. The proofs, which are merely sketched briefly, depend upon certain known theorems concerning functional transformations.

L. A. MacColl.

Bunickij, Evžen. Remarque à l'article "Sur l'intégration des différentielles totales." Časopis Pěst. Mat. Fys. **72**, 131–136 (1947). (Czech. French summary)

The paper in question appeared in the same Časopis **57**, 87–94 (1928).

Chandrasekhar, S. The softening of radiation by multiple Compton scattering. Proc. Roy. Soc. London. Ser. A. **192**, 508–518 (1948).

The problem considered by the author is that of the effect of the Compton scattering in transmission through an atmosphere of free electrons on the radiation of a particular wave length, under the assumption that an infinite plane surface radiates uniformly in the outward direction with a known spectral distribution and that above such a radiating surface there is an atmosphere of free electrons. Mathematically this problem reduces to a boundary value problem of the elliptic equation $f_{xx}+f_{yy}-f=0$, with the boundary conditions $[f_x+3^{\frac{1}{2}}(f+f_y)]_{x=x_1}=2e^{-y}\psi(y)$ and $[f_x-3^{\frac{1}{2}}(f+f_y)]_{x=x_1}\equiv 0$, where $\psi(y)$ is a known function of the specific intensity of radiation of the wave length y. The solution of the boundary value problem is based on an application of Green's theorem.

A. Gelbart (Syracuse, N. Y.).

Pinl, M. Zur Integrationstheorie adiabatisch-kompressibler Potentialströmungen. Z. Angew. Math. Mech. **25/27**, 153–154 (1947).

This note is a brief summary of some work by M. Eichler on the integration of the linear partial differential equations of the plane motion of a circulation free, adiabatic gas. By a suitable change of variables the equation for the stream function takes the form

$$\Delta u+M(x, y)(u_x+iu_y)+N(x, y)u=0. \tag{1}$$

A solution of this equation can be represented in the form

$$f(z)-\int_0^z S(x, y, \zeta)f(\zeta)d\zeta=L[f(z)],$$

where $f(z)$ is an arbitrary analytic function and $S(x, y, \zeta)$ satisfies the following three conditions: (a) $S(x, y, \zeta)$ is analytic in ζ over the same region as z, (b) $S(x, y, \zeta)$ is a particu-

lar integral of the equation (1), (c) when $S(x, y, x+iy)=T$, $T_x+iT_y=\frac{1}{2}N(x, y)$, so that associated to every analytic function $f(z)$ there is a solution of (1). Explicit forms of the generating function S are given for explicit forms of equation (1); in particular for the gas dynamical equations S is given in terms of an infinite series of particular solutions of these equations. *A. Gelbart* (Syracuse, N. Y.).

Švec, M. E. On the heating of an inhomogeneous rod. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 213–214 (1948). (Russian)

The partial differential equations

$$\frac{\partial T_i}{\partial t}=\frac{\partial}{\partial z}\left(b_i z^{n_i}\frac{\partial T_i}{\partial z}\right), \quad i=1, 2, \quad 0\leqq n_i<1,$$

satisfying the boundary conditions

(a) $T_1(t, 0)=T_2(t, 0), \quad \lambda_1\partial T_1/\partial z-\lambda_2\partial T_2/\partial z=S(t),$

(b) $T_1(z, 0)=\phi_1(z), \quad T_2(z, 0)=\phi_2(z),$

are discussed. The substitution $T_i=T_i^{(1)}+T_i^{(2)}$, $i=1, 2$, where

(c) $$T_i^{(1)}(z, 0)=\phi_i(z), \quad \lim_{z\to 0}\lambda_i\partial T_i^{(1)}(z, t)/\partial z=0,$$
$$T_i^{(2)}(z, 0)=0, \quad \lim_{z\to 0}\lambda_i\partial T_i^{(2)}(z, t)/\partial z=Q_i,$$

leads to the determination of $T_i^{(2)}(z, t)$ in terms of Q_i:

$$T_i^{(2)}(z, t)=A_i\int_0^t \frac{Q_i(a)}{(t-a)^{p_i}}\exp\left(\frac{-p_i^2 z^{1/p_i}}{b_i(t-a)}\right)da,$$

where

$$A=p_i^{2p_i-1}/b_i\Gamma(c_i+1), \quad p_i=\tfrac{1}{2}-n_i, \quad i=1, 2,$$

(b_i, c_i constants). Application of boundary condition (b) leads to an Abel-type integral equation

(d) $$\int_0^t Q_i(a)\{c_1(t-a)^{-p_1}+c_2(t-a)^{-p_2}\}da=F_i(t), \quad i=1, 2,$$

where $F_i(t)$ is known. If $p_1=p_2$, the solution may be obtained by well-known methods. For the case $p_1\neq p_2$ the author refers to Whittaker [Proc. Roy. Soc. London. Ser. A. **94**, 367–383 (1918)]. *R. Bellman.*

Dacev, A. B. On the linear problem of Stephan. Doklady Akad. Nauk SSSR (N.S.) **58**, 563–566 (1947). (Russian)

The author considers the partial differential equations

$$a_1^2\partial^2 u_1/\partial x^2=\partial u_1/\partial t, \quad -\infty\leqq x<s,$$
$$a_2^2\partial^2 u_2/\partial x^2=\partial u_2/\partial t, \quad s<x\leqq\infty,$$

subject to $u_1(x, t_0)=\psi_0(x)$, $u_2(x, t_0)=\phi_0(x)$, and, along the curve $x=s(t)$, $ds/dt=k_1\partial u_1/\partial x-k_2\partial u_2/\partial x$. The general problem is as yet unsolved. The author extends, using a different method, however, a particular case treated by Rubinstein [Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **11**, 37–54 (1947); these Rev. **8**, 517]. *R. Bellman* (Stanford University, Calif.).

Smolickiĭ, H. L. On almost-periodic generalized solutions of the wave equation. Doklady Akad. Nauk SSSR (N.S.) **60**, 353–356 (1948). (Russian)

Let us consider the wave equation

$$\Box u=u_{xx}+u_{yy}+u_{zz}-u_{tt}=0$$

subject to the condition $u=0$ on S, the boundary of a finite region $R(x, y, z)$. Let E be the space of functions $\phi(R, t)$ continuous in $\bar{R}\times t$, $\bar{R}=R+S$, $-\infty<t<\infty$, and equal to zero outside of some bounded interval $t_1\leqq t\leqq t_2$, dependent on ϕ. Let $\bar{E}$ be the set of all continuously twice differentiable functions in (R, t) belonging to E. The set in $\bar{E}$ satisfying $u=0$ on S is called E_1, that satisfying $\partial u/\partial n=0$ on S is called E_2. Let E^* be the space of linear functionals ρ on E, and (ρ, ϕ) the value of the functional applied to $\phi\epsilon E$. A functional $\rho\epsilon E^*$ is called a generalized solution of type I (II) if $(\rho, \Box\phi)=0$ for all $\phi\epsilon E_1$ (E_2).

Let $w(x)$ be an even function, continuously differentiable to all orders, with the further properties that $w(x)=1$, $|x|\leqq\frac{1}{2}$; $w(x)=0$, $|x|\geqq 1$; and $w(x)$ is monotone in $[\frac{1}{2}, 1]$; set $k=\int_{-1}^{1}w(x)dx$. The functional ρ_h defined by

$$(\rho_h, \phi)=\left(\rho, (kh)^{-1}\int_{t-h}^{t+h}w((a-t)/h)\phi(R, a)da\right)$$

is called the mean value of the functional ρ with the weight $w(x)$. If $\rho\epsilon E^*$ has the property that

$$(\rho, \phi)=\int_t\int_R \rho(R, t)\phi(R, t)dRdt$$

then $\rho(R, t)$ is called the kernel of the functional ρ. Previous results by Sobolev [C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 542–545, 618–620 (1945); **49**, 12–15 (1945); these Rev. **8**, 78] had considered the case $u=0$ on S. The author considers the condition $\partial u/\partial n=0$ on S, and proves the following theorem. If ρ is a generalized solution of type II, then ρ_h has a kernel $\rho_h(R, t)$ having continuous derivatives of all orders. The mean value of the functional ρ_h has a kernel $\rho_{hh}(R, t)$, twice continuously differentiable, satisfying the wave equation and the boundary condition $\partial u/\partial n=0$ on S. The almost-periodic property of the solution is derived using results contained in Sobolev's papers quoted above. *R. Bellman* (Stanford University, Calif.).

Bureau, Florent. Sur le problème de Cauchy pour les équations linéaires aux dérivées partielles totalement hyperboliques, à un nombre impair de variables indépendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 587–610 (1947).

Bureau, Florent. Sur le problème de Cauchy pour les équations linéaires aux dérivées partielles totalement hyperboliques, à quatre variables indépendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 684–711, 827–853 (1947).

Abstracts of these papers appeared previously [C. R. Acad. Sci. Paris **225**, 852–854 (1947); **226**, 150–152 (1948); these Rev. **9**, 441]. Let $f(\partial/\partial x, \partial/\partial y, \cdots, \partial/\partial t)u=0$ be a differential equation with constant coefficients, $f(\alpha, \beta, \cdots, \tau)$ denoting a form of degree n. The author considers the problem of determining u from its Cauchy data on the hyperplane $t=0$. The equation is assumed to be totally hyperbolic, and the plane to be space-like; i.e., the characteristic surface $f(\alpha, \beta, \cdots, 1)=0$ shall consist of $n/2$ distinct ovals ω_k ($k=1, \cdots, n/2$), where ω_k is contained in ω_{k+1}, and ω_1 contains the origin.

An elementary solution v is constructed in terms of integrals involving the characteristic surface and its intersections with the plane $E=x\alpha+y\beta+\cdots+t=0$. In the first paper this construction is carried out for the case of 3 independent variables. In that case v takes the form

$$v(x, y, t)=i\pi\sum_{k=1}^{n}\epsilon_k\int_{(\alpha_0, \beta_0)}^{(\alpha_k, \beta_k)}E^{n-3}f_\beta^{-1}d\alpha,$$

where (α_0, β_0) is an arbitrary point, (α_k, β_k) are the points

The concluding remarks of the review apply only to the case of an odd number p of independent variables. In the case of an even p the "auxiliary solution" obtained by Herglotz differs essentially from the "elementary solution" of Bureau, which is needed for the solution of the Cauchy problem. In the case of an even p the solution of the Cauchy problem involves the use of the "logarithmic part" of an integral, instead of Hadamard's "finite" part, which is used for odd p. Erratum, p. 855, v. 10, Math. Rev.

of intersection of the characteristic curve with the line $E=0$, ϵ_k denotes sgn $[\Re(\beta_k/i\alpha_k)]$ and f_2 stands for $\partial f/\partial\beta$. In the second paper the case of 4 independent variables is taken up. In that case v is given the form

$$v(x, y, z, t) = \int_{\omega_k} E^{n-4} \log E \, d\omega,$$

where the sum is extended over all k for which the plane $E=0$ intersects the oval ω_k, and where $d\omega$ denotes a certain invariant element of surface. (The ω_k here are assumed to be convex.)

These elementary solutions appear to be essentially the same as those found by G. Herglotz for equations in any number p of independent variables for $n \geqq p$ [see Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **78**, 93–126, 287–318 (1926); **80**, 69–114 (1928); Abh. Math. Sem. Hamburgischen Univ. **6**, 189–197 (1928)].

To derive the solution of the Cauchy problem the author applies Green's formula to the functions u and v, using Hadamard's device of "finite parts of an integral" to handle the divergent expressions arising on the surface of the characteristic cone. *F. John* (New York, N. Y.).

Difference Equations, Special Functional Equations

***Fort, Tomlinson. Finite Differences and Difference Equations in the Real Domain.** Oxford, at the Clarendon Press, 1948. vii+251 pp. $8.00.

We quote from the author's preface. "A selection of material was necessary in order to bring the work within reasonable compass and also to follow somewhat the special interests of the author. The very large theory of difference equations in the field of analytic functions of a complex variable has been completely omitted." ... "The author believes that the treatment of the linear recurrent relation ... will be found particularly interesting. In fact this constitutes the major portion in which most of the original work is located." The book contains many well chosen exercises, enhancing its value for use in a course on difference equations.

In chapter I are given some elementary formulas relating to the direct difference operator. Chapter II concerns the elementary aspects of solving the "summation" equation $\Delta\phi(x) \equiv \phi(x+h) - \phi(x) = f(x)$. Chapters III–V relate to Bernoulli and Euler polynomials, Stirling numbers, numerical differentiation, etc. Chapter VI (whose first draft was due to J. A. Shohat) covers interpolation and mechanical quadratures (Gauss's, Stirling's, Bessel's, Cotes's formulas; various rules of quadrature; Chebyshev's formula). In chapter VII are presented some of the simpler aspects of the linear recurrent relation. Chapter VIII contains the problem of finding y so that $\sum_a^{b-1} F(x_i, y, y(x_{i+1}) - y(x_i))$ is a maximum or minimum. Chapters IX and X refer to the "two-point" boundary problem for linear equations, with coefficients defined for $i = a, \cdots, b-1$, and some theorems of oscillation and comparison. A study of passage to the limit from a difference to a differential equation, as applying to Sturm-Liouville boundary problems, is found in chapter XI. In chapter XII the author studies the problem of the weighted vibrating string [Johann Bernoulli]; this amounts to a Sturm-Liouville problem. Linear recurrent relations with periodic coefficients, sets of functions orthogonal in a suitable sense and corresponding expansions are presented in chapters XIII–XV [see the author's work in Amer. J. Math. **37**, 43–54 (1915); **39**, 1–26 (1917); Quart. J. Math. **45**, 239–257 (1914); **46**, 1–13 (1915)]. A study of oscillatory and nonoscillatory second order equations in chapter XVI represents a continuation of the preceding chapters. Finally, in chapter XVII some of the simpler results are given for the nth order linear equation, with a continuous independent variable. *W. J. Trjitzinsky* (Urbana, Ill.).

Negoescu, Nicolas. Sur une équation de récurrence triangulaire. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **2**, 93–97 (1947).

Solutions in the form of sums involving minors of determinants of binomial coefficients are given for the followng recurrence relation and its instances:

$$a(n+1, p) = (p+i)a(n, p) + (p+i+j)a(n, p-1)$$

with boundary relations $a(0, 0) = 1$, $a(0, p) = 0$, $p \neq 0$ (which entails a triangular array of values); but this has the simple solution $a(n, p) = \binom{p+i+j}{p} \Delta^p i^n$ in the usual notation of finite differences. With $j = -1$, as noted by the author, the $a(n, p)$ appear as coefficients in the expansion of the nth iterate of Milne-Thomson's operator Π_1 [The Calculus of Finite Differences, Macmillan, London, 1933, p. 439] in powers of the displacement operator E. *J. Riordan* (New York, N. Y.).

Aczél, J. Über eine Klasse von Funktionalgleichungen. Comment. Math. Helv. **21**, 247–252 (1948).

It is well known that the only continuous solutions of the Cauchy functional equation

$$f(x_1+x_2) = f(x_1) + f(x_2) \tag{1}$$

are linear functions of the form $f(x) = ax$, and that the only continuous solutions of

$$f(q_1x_1+q_2x_2) = q_1f(x_1) + q_2f(x_2), \tag{2}$$

for fixed positive q_1, q_2 satisfying $q_1+q_2=1$, are linear functions $f(x) = ax+b$. The author now investigates equation (2) with $q_1+q_2 \neq 1$, the equation

$$f(\alpha_1x_1+\alpha_2x_2) = \beta_1f(x_1) + \beta_2f(x_2), \tag{3}$$

and so on.

Thus it is shown that if for all x_1, x_2 we have

$$f(\alpha_1x_1+\alpha_2x_2) = \beta_1f(x_1) + \beta_2f(x_2) + p_1(x_1) + p_2(x_2) + p_0, \tag{4}$$

with $p_1(0) = p_2(0) = 0$, then the function $F(x) = f(x) - f(0)$ satisfies (1), so that the only continuous solutions of (4) are of the form $f(x) = ax+b$. The determination of solutions of (3) under suitable restrictions on α_1, α_2, β_1, β_2 follows. An analogous discussion of mean value functional equations is given. *E. F. Beckenbach* (Los Angeles, Calif.).

Ghermănescu, M. M. Sur quelques extensions de l'équation fonctionnelle de Cauchy. Acad. Roum. Bull. Sect. Sci. **28**, 197–200 (1945).

The general measurable solution of

$$f(x_1+\cdots+x_n) = f(kx_1)+\cdots+f(kx_n)$$

is $f(x) = ax$, with $a=0$ unless $k=1$. Several analogues and generalizations are also studied. *R. P. Boas, Jr.*

Hopf, Eberhard. Über die Funktionalgleichungen der trigonometrischen und hyperbolischen Funktionen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 167–173 (1947).

Concerning the equation $f(x+x') = f(x)+f(x')$ it was shown by Hamel [Math. Ann. **60**, 459–462 (1905)] that a

discontinuous solution must have its graph everywhere dense in the plane. The author proves analogously that for $f(x+x')=f(x)+f(x')$ (mod 1) a solution is either $f(x)\equiv cx$ (mod 1) or has a graph dense on the (x, y)-cylinder (y reduced modulo 1). For $F(x+x')=F(x)F(x')$, a solution is either e^{ax}, or has its graph dense in the plane, or has its graph dense on a set $a\log r+b\varphi+cx=0$ of the (x, y)-plane, with $y=re^{i\varphi}$. Similar results are given for

$$C(x+x')+C(x-x')=2C(x)C(x').$$

R. P. Boas, Jr. (Providence, R. I.).

Montel, Paul. Sur des équations fonctionnelles caractérisant les polynomes. C. R. Acad. Sci. Paris **226**, 1053–1055 (1948).

Let $\Delta_h[f]=f(x+h)-f(x)$. If the continuous function $f(x)$ satisfies $\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}=0$ and $\Delta_{k_1}\Delta_{k_2}\cdots\Delta_{k_n}=0$, with all ratios h_j/k_j irrational, then $f(x)$ is a polynomial of degree at most $n-1$. If the analytic function $f(z)$ satisfies $\Delta_{h_1}\cdots\Delta_{h_n}=0$, $\Delta_{k_1}\cdots\Delta_{k_n}=0$, $\Delta_{l_1}\cdots\Delta_{l_n}=0$, where there is no linear relation with integral coefficients among any triple (h_j, k_j, l_j), then $f(z)$ is a polynomial of degree at most $n-1$. Analogous results are stated for functions of two variables. No proofs are given. *R. P. Boas, Jr.*

Functional Analysis, Ergodic Theory

Haefeli, Hans Georg, und Pellegrino, Franco. Über die Stetigkeit der analytischen Funktionale. Comment. Math. Helv. **21**, 225–246 (1948).

Les auteurs complètent l'etude des fonctionnelles analytiques $F[y(t)]$. Une fonctionnelle analytique est caractérisée par la condition que $F[y(t, \alpha)]=f(\alpha)$ soit une fonction analytique du paramètre α en même temps que $y(t, \alpha)$ [cf. L. Fantappiè, Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 617–706 (1942); ces Rev. **8**, 158]. L'argument $y(t)$ est une fonction analytique d'une ou plusieurs variables; le voisinage de $y_0(t)$ est défini par $|y(t)-y_0(t)|<\sigma$. Pour que $F[y(t)]$ soit continue en $y(t)$, il faut et il suffit qu'elle le soit sur les "droites" $y(t)=y_0(t)+\alpha[y_1(t)-y_0(t)]$ et que $|F[y(t)]-F[y_0(t)]|<\epsilon$ pour $|\alpha|<\delta(\epsilon)$ avec $\min\delta(\epsilon)>0$. Les auteurs démontrent que toute fonctionnelle linéaire analytique est continue. La continuité uniforme est définie et étudiée sur quelques exemples. *P. Lelong.*

Krein, M. The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 431–495 (1947). (Russian. English summary)

The author investigates the set of all self-adjoint extensions $\tilde{T}$ of a semi-bounded (from below) closed Hermitian transformation T, with $\overline{\mathfrak{D}T}=\mathfrak{H}$, $m(T)=\inf(Tf, f)/(f, f)>-\infty$ for $f\in\mathfrak{D}T$, $f\neq 0$ in Hilbert space $\mathfrak{H}$. Their existence (with the same m) was proved by M. H. Stone [Linear Transformations in Hilbert Space, Amer. Math. Soc. Colloquium Publ., v. 15, New York, 1932, theorem 9.21] and by K. Friedrichs [Math. Ann. **109**, 465–487, 685–713 (1934)] and H. Freudenthal [Nederl. Akad. Wetensch., Proc. **39**, 832–833 (1936)]. A new proof is given. The problem is reduced to a positive transformation S, with $\overline{\mathfrak{D}S}=\mathfrak{H}$, and this to generalized bounded closed Hermitian A, with $\mathfrak{D}A=\overline{\mathfrak{D}A}$, by putting $A=(I-S)(I+S)^{-1}$. The transformation A is called Hermitian if $(Af, g)=(f, Ag)$ for every f, g in the domain $\mathfrak{D}A$ of A, even if $\mathfrak{D}A\neq\mathfrak{H}$; this is equivalent to the relation $A\subset A$, as defined by F. J. Murray [Trans. Amer. Math. Soc. **37**, 301–338 (1935)], where A is a transformation "from $\mathfrak{H}$ into $\mathfrak{H}$." If H, H_1 are bounded self-adjoint ($\mathfrak{D}H=\mathfrak{D}H_1=\mathfrak{H}$), $H\leqq H_1$ means $(Hf, f)\leqq(H_1f, f)$ for $f\in\mathfrak{H}$. If $\mathfrak{R}$ is a closed subspace of $\mathfrak{H}$ and $H\neq 0$, as above, with $H\geqq 0$, $H_{\mathfrak{R}}$ denotes $H^{\frac{1}{2}}P_{\mathfrak{L}}H^{\frac{1}{2}}$, where $\mathfrak{L}$ is the subspace composed of all the vectors f for which $H^{\frac{1}{2}}f\in\mathfrak{R}$ (here $H^{\frac{1}{2}}\geqq 0$ and $P_{\mathfrak{L}}$ is the projection on $\mathfrak{L}$).

The fundamental theorem is as follows. For every A with norm $\|A\|=1$ there exist two self-adjoint extensions A_m, A_M with norm 1 and with $\mathfrak{D}A_m=\mathfrak{D}A_M=\mathfrak{H}$, such that a bounded self-adjoint $\tilde{A}$ with $\mathfrak{D}\tilde{A}=\mathfrak{H}$ is an extension of A with $\|\tilde{A}\|=1$ if and only if $A_m\leqq\tilde{A}\leqq A_M$; we have $A_m=\tilde{A}-(I+\tilde{A})_{\mathfrak{R}'}$, $A_M=\tilde{A}+(I-\tilde{A})_{\mathfrak{R}'}$, where $\mathfrak{R}'=\mathfrak{H}\ominus\mathfrak{D}A$. The proof is based on the following lemma. For H and $\mathfrak{R}$ as above, the set of all bounded self-adjoint C, with $\mathfrak{D}C=\mathfrak{H}$, $C\leqq H$ and with the range $\mathfrak{R}C\subset\mathfrak{R}$, contains the transformation $H_{\mathfrak{R}}$ which is "maximal" in the sense $C\leqq H_{\mathfrak{R}}$. The $H_{\mathfrak{R}}$ is studied explicitly. In the proofs, throughout the paper, frequent use is made of the identity $(Hf, f)=\|H^{\frac{1}{2}}f\|^2$ and of generalized orthometrics introduced by Friedrichs [loc. cit.]. Define $\mathfrak{D}[T]$ as the set of all $f\in\mathfrak{H}$ for which there exists a sequence $\{f_n\}$ in $\mathfrak{D}T$ with $f_n\to f$ and $(T(f_n-f_m), f_n-f_m)\to 0$ for $n, m\to\infty$. Then for $f, g\in\mathfrak{D}[T]$, $T[f, g]=\lim(Tf_n, g_n)$ is independent of the choice of $\{f_n\}$ and $\{g_n\}$ [Friedrichs, loc. cit.] and $\mathfrak{D}[T]$ is a complete Hilbert space with the orthometric $(g, f)_T=(g, f)+T[g, f]$. For S, if we put $S_m=(I-A_m)(I+A_m)^{-1}$, and analogously for S_M, then for every positive self-adjoint extension $\tilde{S}$ of S and every $a>0$ we have $(S_m+aI)^{-1}\leqq(\tilde{S}+aI)^{-1}\leqq(S_M+aI)^{-1}$, $\mathfrak{D}[S_m]\subset\mathfrak{D}[\tilde{S}]\subset\mathfrak{D}[S_M]$. A characteristic property is given for S_m, and, in the case $m(S)>0$, also for S_M. This case is studied explicitly and there are theorems like $\mathfrak{D}S_M=\mathfrak{D}S+\mathfrak{N}_0$, $\mathfrak{D}[S_M]=\mathfrak{D}[S]+\mathfrak{N}_0$, $\mathfrak{D}S^*=\mathfrak{D}S_m+\mathfrak{N}_0$, where $\mathfrak{N}_0$ is the set of all roots φ of $S^*\varphi=0$.

The case of a half-bounded transformation with finite deficiency index is studied, and also Hermitian H with "gaps." If H is Hermitian, $\overline{\mathfrak{D}H}=\mathfrak{H}$, then (a, b) is called a gap if $\|Hf-\frac{1}{2}(b+a)f\|\geqq\frac{1}{2}(b-a)\|f\|$ for $f\in\mathfrak{D}H$. J. W. Calkin's results [Duke Math. J. **7**, 504–508 (1940); these Rev. **2**, 224] are completed. If $\lambda_1<\lambda_2<\cdots<\lambda_s$ are regular points (λ "regular" means that there exists at least one self-adjoint extension whose resolvent set contains λ), and $p_1, \cdots, p_s$ are natural numbers with $p_1+\cdots+p_s=\mathfrak{e}(H)$ (where $\mathfrak{e}, \mathfrak{e}$) is the deficiency index of H), then there exists at least one self-adjoint extension of H for which λ_j ($j=1, \cdots, s$) are characteristic values with multiplicity at least p_j. Transformations with discrete spectrum are studied. An important remark [top of p. 437] is incorrect, and, therefore, the proofs of several integral, inf. and sup. theorems by the author do not appear sufficiently clear to the reviewer.

O. M. Nikodým (Gambier, Ohio).

Krein, M. G. The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. II. Mat. Sbornik N.S. **21(63)**, 365–404 (1947). (Russian)

[Cf. the preceding review.] Self-adjoint boundary problems in the differential form

$$(1)\quad f^{[2n]}\equiv p_nf-$$

$$-\frac{d}{dx}\left\{p_{n-1}f'-\frac{d}{dx}[p_{n-2}f''-\cdots-\frac{d}{dx}[p_1f^{(n-1)}-\frac{d}{dx}(p_0f^{(n)})]\cdots]\right\}$$

are investigated, in the direction followed by M. H. Stone [Linear Transformations in Hilbert Space, Amer. Math. Soc. Colloquium Publ., v. 15, New York, 1932], where $p_i(x)$ are real-valued measurable functions in (a, b) such that $\int_a^b dx/|p_0(x)|<\infty$, $\int_a^b |p_k(x)|dx<\infty$ $(k=1, \cdots, n)$. If p_i are sufficiently regular, (1) becomes

$$\sum_{k=0}^{n}(-1)^k\frac{d^k}{dx^k}\left(p_{n-k}\frac{d^kf}{dx^k}\right),$$

to which form the general linear Lagrange-self-adjoint differential classical expression of even order can be reduced [Jacobi, Bertrand]. To manage (1) by Hilbert-space methods put $f^{[0]}=f$, $f^{[k]}=df^{[k-1]}/dx$ $(k=0, \cdots, n-1)$, $f^{[n]}=p_0\cdot df^{[n-1]}/dx$, $f^{[n+k]}=-df^{[n+k-1]}/dx+p_kf^{[n-k]}$ $(k=1, \cdots, n)$ and define D^* as the set of all complex $f\in L_2(a, b)$ for which $f^{[0]}, \cdots, f^{[2n-1]}$ are absolutely continuous in $a\leqq x\leqq b$ and $f^{[2n]}\in L_2(a, b)$. For any $h\in L_2(a, b)$ and any $0\leqq x_0\leqq 1$ the system $f^{[2n]}=h(x)$, $f^{[k]}(x_0)=C_k$ $(k=0, \cdots, 2n-1)$, where C_k are given complex constants, admits one and only one solution $f\in D^*$. Define the operator T in the Hilbert space $L_2(a, b)$, on the set D of all $f\in D^*$ with $f^{[k]}(a)=f^{[k]}(b)=0$ $(k=0, \cdots, 2n-1)$, by putting $Tf=f^{[2n]}$. It is Hermitian with deficiency index $(2n, 2n)$, its domain D is everywhere dense in L_2, and the domain of its adjoint T^* is D^*. Self-adjoint extensions $\tilde{T}$ of T are studied, in close relation with J. W. Calkin's paper [Trans. Amer. Math. Soc. 45, 369–442 (1939)], by attaching the vector $\xi=(\xi_1, \cdots, \xi_{4n})$ in the $4n$-dimensional unitary space $\mathfrak{E}$ to the boundary condition $\xi_k=f^{[k-1]}(a)$, $\xi_{n+k}=f^{[k-1]}(b)$ $(k=1, \cdots, 2n)$, for $f\in D^*$. It is shown that there is a one-to-one correspondence between (1′) self-adjoint homogeneous boundary conditions, (2′) self-adjoint extensions $\tilde{T}$ and (3′) subspaces $\Pi(\tilde{T})$ of $\mathfrak{E}$ such that Π is the orthogonal complement of $U(\Pi)$, where U is the unitary transformation in $\mathfrak{E}$ carrying the vector $(\xi_1, \cdots, \xi_{4n})$ into $(\eta_1, \cdots, \eta_{4n})$; here $\eta_k=\xi_{2n+k-1}$, $\eta_{n+k}=-\xi_{n-k+1}$, $\eta_{2n+k}=-\xi_{4n-k+1}$, $\eta_{3n+k}=\xi_{3n-k+1}$ $(k=1, \cdots, n)$. If (2) $p_0(x)\geqq 0$ in $a\leqq x\leqq b$, then T is half-bounded from below [see the review of part I]. It is stated without proof that the converse also holds true. In case (2) every $\tilde{T}$ has a discrete spectrum. In the case where 0 does not belong to the spectrum the corresponding Green's function $G(x, \xi)$ is built up by means of the Cauchy-function for (1) (i.e., $\Phi(x, \xi)$ which, considered as a function of x, is the solution of $y^{[2n]}=0$ with $\Phi^{[k]}(x, \xi)|_{x=\xi}=0$ $(k=0, \cdots, 2n-2)$, and $\Phi^{[2n-1]}(x, \xi)|_{x=\xi}=1$).

The author extends to the present case some of his results obtained in earlier papers [Rec. Math. [Mat. Sbornik] N.S. 2(44), 1023–1072 (1937); C. R. Acad. Sci. Paris 200, 797–799 (1935)]. The function $G(x, \xi)$ is Hermitian symmetric, and under (2), Mercer's expansion

$$G(x, \xi)=\sum_{\nu=1}^{\infty}\varphi_\nu(x)\overline{\varphi_\nu(\xi)}/\lambda_\nu,$$

where $\lambda_1\leqq\lambda_2\leqq\cdots$ and $\varphi_1, \varphi_2, \cdots$ are orthonormal characteristic elements of $\tilde{T}$, can be differentiated termwise for $\partial^{j+k}/\partial x^j\partial\xi^k$ $(j, k=0, \cdots, n-1)$, obtaining absolutely and uniformly convergent series. Under (2) let $\tilde{T}_\omega$ be the self-adjoint extension of T defined by $f^{[k]}(a)=f^{[k]}(b)=0$ $(k=0, \cdots, n-1)$. Then $\tilde{T}_\omega=T_\mu$ [see the preceding review] and $D[T]$ coincides with the set of all the $f\in L_2$ for which (1″) $f^{[k]}$ is absolutely continuous, (2″) $f^{[k]}(a)=f^{[k]}(b)=0$ $(k=0, \cdots, n-1)$, and (3″) $\int_a^b p_0(x)|f^{(n)}(x)|^2dx<\infty$. Boundary conditions having the form $\sum_{k=1}^{n}\alpha_kf^{[k-1]}(a)+\sum_{k=1}^{n}\beta_kf^{[k-1]}(b)=0$ are termed "principal." For $f\in L_2$ and for $\tilde{T}$ the following are equivalent: (1) $f\in D[\tilde{T}]$, (2) $\sum_{j=1}^{\infty}\lambda_j|(f, \varphi_j)|^2<\infty$, (3) f is almost everywhere equal to an absolutely continuous function f_0, with $f_0^{(k)}$ $(k=1, \cdots, n-1)$ absolutely continuous, satisfying all the principal boundary conditions corresponding to $\tilde{T}$ and with $\int_a^b p_0(x)|f_0^{(n)}(x)|^2dx<\infty$. The "strongly positive" case of T (that is, $T[f, f]>0$ for $0\neq f\in D[T]$) is studied. This case holds when, e.g., all the $p_k(x)\geqq 0$.

There are given inequalities and distribution theorems for the characteristic values of $\tilde{T}$, and also theorems allowing one to calculate the precise number of the negative ones by means of some Hermitian quadratic forms [for classical features see H. Weyl, Math. Ann. 71, 441–479 (1912) and the author's paper of 1937 cited above], e.g., if T is strongly positive, $\tilde{T}$ its self-adjoint extension and $G(x, s)$ the Green's function of $\tilde{T}$, then the number of negative characteristic values of $\tilde{T}$ is equal to the number of negative squares for the form

$$\sum_{j,k=0}^{n}G_{j,k}(a, a)\xi_j\bar{\xi}_k+2\Re\left\{\sum_{j,k=0}^{n-1}G_{j,k}(a, b)\xi_j\bar{\xi}_k\right\}+\sum_{j,k=0}^{n-1}G_{j,k}(b, b)\eta_j\bar{\eta}_k,$$

where $G_{j,k}=\partial^{j+k}G/\partial x^j\partial s^k$. An extension for the case $f(x)-\lambda\int_a^b G(x, \xi)f(\xi)d\sigma(\xi)=0$ is indicated.

O. M. Nikodým (Gambier, Ohio).

Julia, Gaston. Sur une propriété caractéristique des systèmes orthonormaux complets; généralisation. C. R. Acad. Sci. Paris 226, 1485–1487 (1948).

If n and p are nonnegative integers and if E_n and E_{n+p} are unitary spaces whose dimension is equal to their subscript, then a necessary and sufficient condition that a subset $\{x_i: i=1, \cdots, n+p\}$ of E_n have the property that $\|x\|^2=\sum_{i=1}^{n+p}|(x_i, x)|^2$, for every x in E_n, is that there exist a complete orthonormal set $\{y_i: i=1, \cdots, n+p\}$ in E_{n+p} such that x_i is the orthogonal projection of y_i onto E_n, $i=1, \cdots, n+p$. *P. R. Halmos* (Chicago, Ill.).

Tseng, Yuan-Yung. Expansions according to an arbitrary system of functions in hyper-Hilbert space. Sci. Rep. Nat. Tsing Hua Univ. 4, 286–312 (1947).

As in the previous work of this author, a unitary space includes those spaces for which the scalars correspond to quaternions. Separability is not assumed. The objective is to provide a suitable generalization of the Gram-Schmidt process and also of the usual properties of the resulting orthonormal series, i.e., "best linear approximation," Bessel's inequality and Parseval's equality. Precise formulas are given in the notation of general analysis. The exact role of completeness in these discussions is shown. The results are applied to the domain of operators A for which A^{**} exists and to the appropriate portions of the Hellinger integral development. *F. J. Murray* (New York, N. Y.).

Mil'man, D. Isometry and extremal points. Doklady Akad. Nauk SSSR (N.S.) 59, 1241–1244 (1948). (Russian)

This note gives a proof of the representation theorem proved by Banach [Théorie des Opérations Linéaires, Warsaw, 1932, p. 173], for Q the interval $0\leqq q\leqq 1$ and by Stone [Trans. Amer. Math. Soc. 41, 375–481 (1937), theorem 83] in more than the present generality. Let Q be a compact Hausdorff space; let C_Q be the normed space of all real-valued continuous functions on Q and let U be an isometric linear transformation of C_Q onto itself. Then there exists an η in C_Q with $|\eta(q)|=1$ for all q and a homeomorphism φ of Q onto itself such that $y=Ux$ if and only if $y(q)=\eta(q)x(\varphi(q))$ for all q in Q. The proof is based on theorem 1: if U is an isometric linear transformation of the Banach space E_1 onto E_2, then U^* is a homeomorphism in the w^*-topology between

the sets of extreme points of the unit spheres in E_2^* and in E_1^*. If, moreover, P_x is the set of f of norm one such that $f(x)=\|x\|$, U^* is also a w^*-homeomorphism of the sets of extreme points of P_{Ux} and P_x. Corollary: each isometric operator from E_1 onto E_2 is an extreme point of the set of linear operators of norm not exceeding 1. To prove the representation theorem, the mapping $q \leftrightarrow f_q$ defined by $f_q(x)=x(q)$ for all x in C_Q is a homeomorphism of Q with the set of extreme points of P_e in the w^*-topology, where e is the function identically one. Then U^* is a w^*-homeomorphism of this set onto the extreme points of P_{Ue}. The fact that every extreme point of the unit sphere of C_Q is either in P_e or $-P_e$ then gives $U^*f_q=\eta(q)f_{\varphi(q)}$ for all q in Q; this η and φ have the desired properties. A similar proof is given independently by Arens and Kelley [Trans. Amer. Math. Soc. 62, 499–508 (1947); these Rev. 9, 291].

M. M. Day (Princeton, N. J.).

Monna, A. F. Sur les espaces linéaires normés. V. Nederl. Akad. Wetensch., Proc. 51, 197–210 = Indagationes Math. 10, 68–81 (1948).

[Parts I, II, III, IV appeared in the same Proc. 49, 1045–1055, 1056–1062, 1134–1141, 1142–1152 = Indagationes Math. 8, 643–653, 654–660, 682–689, 690–700 (1946); these Rev. 9, 43.] Let K be a field complete with respect to a nontrivial, non-Archimedean valuation. Let E be a linear normed space over K satisfying $\|x+y\| \leqq \max(\|x\|, \|y\|)$; assume that E is complete and separable, and that the norms $\|x\|$ have only 0 as limit point. The vector x is said to be pseudo-orthogonal to y if $\|x-ay\| \geqq \|x\|$ for all $a \in K$; this relation is symmetric. It is proved that there exists a finite or countable pseudo-orthogonal sequence $\{x_i\}$ such that every vector is uniquely of the form $\sum_i a_i x_i$, $a_i \in K$. The space E is topologically isomorphic to a space of m-tuples $(a_1, \cdots, a_m)$ or of sequences $\{a_i\}$ with $\lim a_i=0$, $a_i \in K$, with norm given by $\max_i |a_i|$. If E is infinite-dimensional, it cannot be isomorphic to its conjugate space, nor can it be reflexive. There is a discussion of strong, weak and coordinatewise convergence. *I. S. Cohen* (Cambridge, Mass.).

Sheffer, I. M. On k-periodic systems of linear equations. Trans. Amer. Math. Soc. 63, 244–313 (1948).

The system of linear equations is the infinite system $\sum_{j=0}^{\infty} a_{nj}x_{n+j}=c_n$, where the matrix $\mathfrak{A}=(a_{nj})$ is essentially a matrix with zeros below the main diagonal. The sequences $X=\{x_n\}$ and $C=\{c_n\}$ are subject to the condition that $\|X\|=\limsup |x_n|^{1/n} \leqq q$ for fixed q. As a consequence the rows of the matrix are assumed to be such that the functions $A_n(t)=\sum_{j=0}^{\infty} a_{nj}t^j$ are analytic for $|t| \leqq q$. The matrix $\mathfrak{A}$ is k-periodic if $a_{n+k,j}=a_{nj}$ for all n and j or $A_{n+k}(t)=A_n(t)$ for all n. The paper is an extension of the case where $k=1$ as treated by O. Perron [Math. Ann. 84, 1–15 (1921)]. Central in the theory are the roots of $\Delta_A(t)$, the kth order determinant of $\omega^n A_n(\omega^m t)$, $m, n=0, 1, \cdots, k-1$, where ω is a primitive kth root of unity, which satisfies the condition $\Delta_A(\omega^r t)=\Delta_A(t)$ for $r=1, \cdots, k-1$, so that it is essentially a function of t^k and its zeros appear in nests: $(\omega^j \alpha)$, $j=0, 1, \cdots, k-1$. If $\Delta_A(t)$ has l nests of zeros for $|t| \leqq q$ then $\mathfrak{A}$ is said to be of order l. The matrix $\mathfrak{A}$ is said to be admissible if the condition that $a_{n0} \neq 0$ for all n is added to the preceding conditions. It is shown that there exist admissible matrices $\mathfrak{B}$ such that if the system is multiplied or transformed via $\mathfrak{B}$ then any solution of the original system is also a solution of the transformed system and the transformed system has period one. To reverse the statement it is shown that if the matrix $\mathfrak{A}$ is of order l it can be factored in the form $\mathfrak{C}_1\mathfrak{B}_1\mathfrak{C}_2\mathfrak{B}_2 \cdots \mathfrak{C}_l\mathfrak{B}_l\mathfrak{C}_{l+1}$, where $\mathfrak{C}_i$ are admissible matrices of order zero and $\mathfrak{B}_i$ are admissible matrices of order one of a canonical type. As a consequence it follows that if $\mathfrak{A}$ is of order l in $|t| \leqq q$ then the homogeneous system $\sum_{j=0}^{\infty} a_{nj}x_{n+j}=0$ has l linearly independent solutions with $\|X\| \leqq q$ and the nonhomogeneous system $\sum_{j=0}^{\infty} a_{nj}x_{n+j}=c_n$ with $\limsup |c_n|^{1/n} \leqq q$ has a general solution with $\|X\| \leqq q$ containing l arbitrary constants. A similar result is obtained for the perturbed system $\sum_{j=0}^{\infty}(a_{nj}+a_{nj}^*)x_{n+j}=c_n$, where the matrix $\mathfrak{A}=(a_{nj})$ is admissible, the matrix $\mathfrak{A}^*=(a_{nj}^*)$ is subject to the condition that there exist positive constants $k_n \downarrow 0$ and $\theta<1/q$ such that $|a_{nj}| \leqq k_n\theta^j$, and $a_{n0}^*+a_{n0} \neq 0$ for all n. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Hopf, Eberhard. Über eine Ungleichung der Ergodentheorie. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1944, 171–176 (1947).

The author simplifies Pitt's proof [Proc. Cambridge Philos. Soc. 38, 325–343 (1942); these Rev. 4, 219] of the following result of Kakutani and Yosida [cf. Proc. Imp. Acad. Tokyo 15, 165–168 (1939); these Rev. 1, 59]. If T is a one-to-one measure-preserving transformation of a finite measure space onto itself, if f is a real-valued integrable function and if

$$E=\left\{x: \sup_{n \geqq 1} n^{-1}\sum_{i=0}^{n-1} f(T^i x)>t\right\},$$

then $\int_E f dm \geqq t m(E)$. *P. R. Halmos* (Chicago, Ill.).

Carathéodory, C. Bemerkungen zum Ergodensatz von G. Birkhoff. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1944, 189–208 (1947).

In the framework of Boolean algebras (instead of measure spaces) the author states and proves a version of the Birkhoff ergodic theorem which applies to not necessarily one-to-one transformations. [In this connection see F. Riesz, Comment. Math. Helv. 17, 221–239 (1945); these Rev. 7, 255. The first proof of the non-one-to-one ergodic theorem, apparently overlooked by most writers on the subject, was given by J. L. Doob, Duke Math. J. 6, 290–306 (1940); these Rev. 1, 344. Cf. in particular theorem 1, p. 291.]

P. R. Halmos (Chicago, Ill.).

Minkevič, M. I. Closed integral funnels in generalized dynamical systems without a hypothesis of uniqueness. Doklady Akad. Nauk SSSR (N.S.) 60, 341–343 (1948). (Russian)

The author continues the study of one-parameter systems of set-valued transformations which he began in an earlier paper [same Doklady (N.S.) 59, 1049–1052 (1948); these Rev. 9, 449]. Let R be a compact metric space with metric ρ. Let C be the collection of all nonvacuous closed subsets of R and let C be metrized by the Hausdorff metric, say σ. Let $S: R \times T \to C$, where T denotes the set of real numbers. Assume that (1) if $p \in R$, then $S(p, 0)=p$; (2) if $p \in R$ and if $t, s \in T$ with $ts \geqq 0$, then $S(S(p, t), s)=S(p, t+s)$; (3) if $p, q \in R$ and $t \in T$ with $q \in S(p, t)$, then $p \in S(q, -t)$; (4) the function $S(p, t)$ ($p \in R$, $t \in T$) is continuous in t for each fixed p. A system of differential equations $x'=f(x)$ (x and $f(x)$ are n-dimensional vectors; $f(x)$ is continuous) defines a transformation S satisfying the above conditions.

Let T^+ (T^-) denote the set of nonnegative (nonpositive) real numbers. If $p \in R$, then the sets $S(p, T)$, $S(p, T^+)$, $S(p, T^-)$ are called the funnel, ω-funnel, α-funnel of p. A point p of R

is called ω-continuous provided that if $t>0$ and if $\epsilon>0$, then there exists $\delta>0$ such that $q\varepsilon R$ with $\rho(p,q)<\delta$ implies $\sigma[S(p,t'),S(q,t')]<\epsilon$ $(0\leqq t'<t)$. The ω-funnel of $p\varepsilon R$ is called closed in case: (1) $S(p,T^+)=S(p,I)$ for some finite interval $I\subset T^+$; (2) if $q\varepsilon S(p,T^+)$, then $p\varepsilon S(q,T^+)$. A non-vacuous closed subset E of R is called ω-stable (in the sense of Poisson) provided that if $\epsilon>0$ and if $t>0$, then there exists $t'\geqq t$ such that $\sigma[E,S(E,t')]<\epsilon$. A point p of R is called ω-recurrent provided that if $\epsilon>0$, then there exists $L>0$ such that $t',t''\varepsilon T^+$ implies $\sigma[S(p,t'),S(p,t''+t)]<\epsilon$ for some t with $0\leqq t\leqq L$. A point p of R is called ω-periodic provided that $S(p,t)=p$ for some $t>0$. The notions of α-continuity, closedness of an α-funnel, etc., are defined dually.

The following theorems, together with their duals, are announced without proof. (1) If $p\varepsilon R$, if the ω-funnel of p is closed and if $t\varepsilon T^+$, then $S(p,t)\subset S(p,t+t')$ for some $t'>t$. (2) If $p\varepsilon R$ and if the ω-funnel of p is closed, then $S(p,t)$ is a periodic function of t for sufficiently large t. (3) If $p\varepsilon R$, if $t\varepsilon T^+$ and if each point of $S(p,t)$ is ω-continuous and ω-stable, then $S(p,t')$ is ω-stable for all $t'>t$. (4) If $p\varepsilon R$ and if p is ω-recurrent, then $S(p,t)$ is ω-stable for all $t\varepsilon T^+$. (5) For a dynamical system within the plane (unique solutions not necessary) every ω-stable point is ω-periodic. (6) If $p\varepsilon R$, if every point of $S(p,T^+)$ is ω-stable, if $q\varepsilon S(p,T^+)$ and if q is ω-periodic, then $S(q,t)$ reduces to a point for all $t\varepsilon T^+$. (7) If $p\varepsilon R$ and if p is α-periodic and ω-periodic, then the α-period of p and the ω-period of p are equal in absolute value. *W. H. Gottschalk* (Charlottesville, Va.).

Gottschalk, W. H. Recursive properties of transformation groups. II. Bull. Amer. Math. Soc. **54**, 381–383 (1948).

Let T be a locally compact transformation group acting continuously on a topological space X. Let there be distinguished in T a class of subsets, called admissible, with the property that if $A\subset BK$ for some compact set K, and A is admissible, then B is admissible. A subgroup R of T is said to be recursive at a point $x\varepsilon X$ if to each neighborhood U of x there corresponds an admissible set A such that $A\subset R$ and $xA\subset U$. It is shown that if T is recursive at x, and S is an invariant subgroup of T such that $T=SK$ for some compact set K, then S is recursive at x. This sharpens a previous result due to the author and G. A. Hedlund [Bull. Amer. Math. Soc. **52**, 637–641 (1946); these Rev. **8**, 34]. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Theory of Probability

Kossack, Carl F. The existence of collectives in abstract space. Sankhyā **8**, 219–234 (1947).

The author defines a randomness which is analogous to yet quite distinct from that required of the collectives of von Mises. He proves the existence of sequences satisfying this condition of randomness. The terms of the sequences are elements of an abstract space S. Associated with S is a denumerable field F and a distribution function defined over F. The author considers the Lebesgue extension of F and the corresponding extension of the distribution function. He requires that the sequences possess this extended distribution function and that they satisfy the conditions of admissibility which assure the independence of the terms. If a sequence is operated upon by any one of an arbitrary denumerable set of place selections the resultant sequence must satisfy the above conditions. If the sequence is operated upon by a contraction (called Teilung by von Mises) the distribution function is altered in a well-defined manner and the resultant sequence must satisfy the above conditions with respect to the altered distribution function. Thus the above properties are invariant under the operations of selection and contraction if the number of such operations applied to a given sequence is finite. A sequence for which this is the case is regarded as random. The author shows that almost every sequence is random in this sense. *A. H. Copeland* (Ann Arbor, Mich.).

Fortet, Robert. Sur la probabilité de perte d'un appel téléphonique. C. R. Acad. Sci. Paris **226**, 1502–1504 (1948).

The results of a preceding note [same C. R. **226**, 159–161 (1948); these Rev. **9**, 361] are generalized to the case where there are n groups of x selectors each and each group is preceded by a unique orienter. *W. Feller.*

Hole, Njål. On the statistical treatment of counter data. Ark. Mat. Astr. Fys. **34B**, no. 20, 8 pp. (1948).

The paper opens with a few general remarks concerning the difference between transient and steady states. The author then discusses maximum likelihood estimates for the means and variances of losses in counters of the two usual types. *W. Feller* (Ithaca, N. Y.).

Kanellos, S. G. Two problems of the calculus of probability. Bull. Soc. Math. Grèce **23**, 132–142 (1948). (Greek. English summary)

Consider Bernoulli trials with possible results A and B. The author calls a succession BAB "an alternation of A." Using the difference equation method he finds the probabilities that n trials result (1) in no A-alternation; (2) in neither an A- nor a B-alternation. *W. Feller.*

Sibirani, Filippo. Esperienze sul comportamento delle frequenze in eventi di probabilità costante ma ignota. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **9**, 167–174 (1942).

Description and analysis of 30 series of 200 observations each of a rather complicated random experiment with a deck of 40 cards. There were 40 possible outcomes of each experiment. *W. Feller* (Ithaca, N. Y.).

Sibirani, Filippo. Alcune probabilità geometriche. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **1**, 113–123 (1944).

A simple proof of Halphen's theorem [Bull. Soc. Math. France **1**, 221–224 (1873)] to the effect that if an interval is divided into n random parts there is probability $1-n/2^{n-1}$ that the segments equal the sides of a closed polygon. For $n=3$ the author finds also the probability that the triangle be acute. The problem is generalized by considering, in addition to the n random parts, m segments of preassigned lengths. *W. Feller* (Ithaca, N. Y.).

de Broglie, Louis. Sur la statistique des cas purs en mécanique ondulatoire. C. R. Acad. Sci. Paris **226**, 1056–1058 (1948).

Continuing his discussion of bivariate probability distributions of the observables of quantum mechanics [same C. R. **223**, 874–877 (1946); these Rev. **8**, 300], the author exhibits the difficulties involved by discussing a simple card

game in which (corresponding to the fact that observables may not be simultaneously measurable, and that the measurement of one observable may influence a second) it is supposed that the observation that the suit of a card is hearts implies that the card is a king and the observation that a card is a king implies that the suit is spades. Sampling under such conditions means that the usual rules for combining conditional probabilities are no longer valid, and in fact the cards lose their objective character, in the sense that, for example, if one observes successively that a card is a heart and a king one cannot say that it is a king of hearts. *J. L. Doob* (Urbana, Ill.).

Bennett, W. R. Distribution of the sum of randomly phased components. Quart. Appl. Math. **5**, 385–393 (1948).

In the general form of the problem of random flights we are required to find the distribution of the sum of n vectors with specified lengths and randomly distributed directional angles. Kluyver's formula for this distribution is

$$P_n(r)=r\int_0^\infty J_1(rs)\prod_{m=1}^{n}J_0(a_m s)\,ds,$$

where $P_n(r)$ is the probability that the sum of n components having individual magnitudes $a_1,\cdots,a_n$ is less than r [Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1922, pp. 419–421]. For $n=1, 2, 3$ the integral is expressible in terms of elementary functions or elliptic integrals. For large values of n the integral may be calculated from asymptotic series. For intermediate values of n (e.g., n of the order of 10) the accuracy of the asymptotic series was open to suspicion, and a better method of evaluating the integral was sought. In the particular case $a_m=1$ $(m=1,\cdots,n)$ the author obtains

$$\varphi_n(k)=1-P_n(k)=1-k\int_0^\infty J_1(ks)J_0^n(s)\,ds$$

$$=2\sum_{m=1}^{\infty}\frac{J_1(j_m/n)J_0^{n-1}(j_m/n)}{j_m J_1^2(j_m)}J_0(kj_m/n),$$

where j_m is the mth root, in ascending order of magnitude, of $J_0(x)=0$. In the general case the result is

$$\varphi_n(r)=\frac{2}{A}\sum_{m=1}^{\infty}\frac{b_m c_m J_0(rj_m/A)}{j_m J_1^2(j_m)},\quad A=\sum_{m=1}^{n}a_m,$$

$$c_m=\sum_{s=1}^{n}a_s J_1(a_s j_m/A)/J_0(a_s j_m/A),\quad b_m=\prod_{s=1}^{n}J_0(a_s j_m/A).$$

[In the formula for b_m in the paper, $\sum$ is printed instead of $\prod$.] The author states a general mathematical theorem of which the above results are special cases.

S. C. van Veen (Delft).

Slack, Margaret. A problem on the summation of simple harmonic functions of the same amplitude and frequency, but of random phase, and the probability distributions of sinusoidal oscillations combined in random phase. Philos. Mag. (7) **38**, 297–298 (1947).

With reference to a paper by Horner [same Mag. (7) **37**, 145–162 (1946); these Rev. **9**, 97] it is pointed out that the density function of $l\sum_1^n\cos\varphi_k$ (φ_k's independent and uniformly distributed in $(0, 2\pi)$) is

$$(l\pi)^{-1}\int_0^\infty\{J_0(l\beta)\}^n\cos\beta s\,ds.$$

The author is unaware of the fact that the formula is obtainable by an application of characteristic functions and the Fourier inversion formula. *M. Kac* (Ithaca, N. Y.).

Gladwin, A. S. Energy distribution in the spectrum of a frequency modulated wave. II. Philos. Mag. (7) **38**, 229–251 (1947).

Extension to a more general situation of results reviewed previously [same Mag. (7) **35**, 787–802 (1944); these Rev. **7**, 99]. *M. Kac* (Ithaca, N. Y.).

Stumpers, F. L. H. M. On a non-linear noise problem. Philips Research Rep. **2**, 241–259 (1947).

Calculation of the spectrum of the noise output of a non-linear device under the assumption that the spectrum of the input noise is rectangular. The author seems to be unaware of the work of Rice [Bell System Tech. J. **24**, 46–156 (1945); these Rev. **6**, 233]. *M. Kac* (Ithaca, N. Y.).

Bobrov, A. A. A simplified proof of a theorem of A. N. Kolmogorov on the strong law of large numbers. Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 194–196 (1947). (Russian)

Consider a sequence of sums $s_n=\xi_1+\cdots+\xi_n$ of independent random variables with $E\xi_n=0$, $E\xi_n^2=b_n$ finite for every n, $B_n=b_1+\cdots+b_n$, $P_{n,m}(\epsilon)=\Pr\{\max_{n\le k\le m}|s_n|/n\ge\epsilon\}$. Kolmogoroff's proof of his classical theorem that $\sum_n b_n/n^2<\infty$ entails $P(\epsilon)=\lim_{n\to\infty}\lim_{m\to\infty}P_{n,m}(\epsilon)=0$ for every given $\epsilon>0$ is simplified by the author as follows. (a) If e_k is the event $|s_{n+i}|\ge\epsilon(n+i)$ for the first time when $i=k$ then $B_{n+p}\ge\epsilon^2\sum_{k=1}^{p}P(e_k)(n+k)^2$, which gives

$$\sum_{p=1}^{m}B_{n+p}\{(n+p)^{-2}-(n+p+1)^{-2}\}$$

$$\ge\epsilon^2\{P_{n,m}(\epsilon)-(n+m+1)^{-2}\sum_{k=1}^{m}P(e_k)(n+k)^2\}.$$

(b) Abel's transformation gives

$$\sum_{p=1}^{m}\frac{b_{n+p}}{(n+p)^2}=\frac{B_{n+m}}{(n+m+1)^2}-\frac{B_n}{(n+1)^2}$$

$$+\sum_{p=1}^{m}B_{n+p}\left[\frac{1}{(n+p)^2}-\frac{1}{(n+p+1)^2}\right].$$

(c) Letting $m\to\infty$, (a) and (b) give

$$\sum_{p=1}^{\infty}b_{n+p}/(n+p)^2+B_n/(n+1)^2\ge\epsilon^2P_{n,\infty}(\epsilon)$$

and, as $\lim_{n\to\infty}B_n/n^2=0$, $P(\epsilon)=0$. *M. Loève.*

Wald, Abraham. On the distribution of the maximum of successive cumulative sums of independently but not identically distributed chance variables. Bull. Amer. Math. Soc. **54**, 422–430 (1948).

Let $M_N=\max S_n$ for $n\le N$, where $S_n=X_1+\cdots+X_n$, the X_k's being independent random variables of the second order. The author is concerned with the properties of M_N and extends his own and Erdös and Kac's results obtained in the particular case of identically distributed X_k's. (a) Lower and upper bounds for M_N are given. (b) The exact probability distribution for M_N is derived when the X_k's can take only the values $+1$ and -1. (c) In the even more general case of two double sequences with corresponding M_N and $\bar{M}_N$, the author proves that, under some

assumptions about means and variances,

$$\liminf_{N\to\infty}[\Pr\{M_N<cN^{\frac{1}{2}}\}-\Pr\{\bar{M}_N<(c-\epsilon)N^{\frac{1}{2}}\}]\geqq 0,$$

$$\liminf_{N\to\infty}[\Pr\{\bar{M}_N<(c+\epsilon)N^{\frac{1}{2}}\}-\Pr\{M_N<cN^{\frac{1}{2}}\}]\geqq 0.$$

M. Loève (Berkeley, Calif.).

Doss, Shafik. Sur la moyenne d'un élément aléatoire abstrait. C. R. Acad. Sci. Paris **226**, 1418–1419 (1948).

Let (x, y) denote the distance $|x-y|$ of the real numbers x and y. The mean $a=E(X)$ of a random variable X is characterized by the property that (*) $(a, t)\leqq E((X, t))$ for all real t. The author notes that (*) retains a sense when X is defined on any metric space with values in another. Thus (*) leads to a general definition of mean for abstract random variables. Several properties are indicated.

W. Feller (Ithaca, N. Y.).

Fréchet, Maurice. Sur une nouvelle définition des positions typiques d'un élément aléatoire abstrait. C. R. Acad. Sci. Paris **226**, 1419–1420 (1948).

The author discusses the relations between the definition of the preceding review and his own definitions [Rev. Sci. (Rev. Rose Illus.) **82**, 483–512 (1944); these Rev. **8**, 141]. He points out that (*) can be used to define not only the mean but other typical values such as moments. He discusses the difficulties. *W. Feller* (Ithaca, N. Y.).

Onicescu, Octav, et Mihoc, Gh. Un cas d'exception dans la théorie des chaînes de Markoff. Acad. Roum. Bull. Sect. Sci. **24**, 401–408 (1943).

Let $x_1, x_2, \cdots$ be the random variables of a Markov chain with stationary transition probabilities. In earlier papers [Bull. Math. Soc. Roumaine Sci. **41**, 99–116 (1939); Mathematica, Cluj **16**, 13–43 (1940); these Rev. **1**, 342] on the distribution of $\sum_1^n x_j$ for large n, studied by means of characteristic functions, the authors were forced to appeal to other methods to handle a certain degenerate case. They now show, deferring the details to a later paper, how their method can be used in all cases. [For a complete treatment of this problem by a different method see W. Doeblin, Bull. Math. Soc. Roumaine Sci. **39**, no. 2, 3–61 (1937).]

J. L. Doob (Urbana, Ill.).

Onicescu, O. Les séries de structures. Acad. Roum. Bull. Sect. Sci. **26**, 503–510 (1946).

Let $x_1, x_2, \cdots$ be mutually independent nonnegative random variables. The author apparently wishes to prove, in the language of "binary structures," that $\sum_1^n x_n$ converges in distribution if and only if the infinite product $\prod_1^n \Pr\{x_n=0\}$ is convergent. The general theorem [which is false] is derived from the special [true] theorem for the case when x_n only takes on the values 0 and 1. *J. L. Doob.*

Bass, Jean. Sur les propriétés des relations fonctionnelles vérifiées par les lois de probabilité des fonctions aléatoires dérivables. C. R. Acad. Sci. Paris **226**, 1120–1122 (1948).

Let $X(t)$ be a random variable depending on the parameter t, with $dX/dt=U(t)$. Then if $f(x, u)$ has bounded second derivatives it is supposed that df/dt satisfies an equation of the form $d\bar{f}/dt=\overline{U\partial f/\partial x}+\overline{Bf}$, where B is a linear operator. The author discusses various hypotheses on B, for example that it be a differential operator, and states that (without assuming the Markov property of the process) suitable continuity hypotheses make B a differential operator of the second order, so that an analogue of the Fokker-Planck equations holds; $d\bar{f}/dt=\overline{U\partial f/\partial x}+\overline{a\partial f/\partial u}+\frac{1}{2}\overline{b^2\partial^2 f/\partial u^2}$, where $a=a(x, u, t)$, $b=b(x, u, t)$ do not depend on x if the $U(t)$ process is a Markov process. The latter hypothesis was made in an earlier paper [C. R. Acad. Sci. Paris **220**, 190–192 (1945); these Rev. **7**, 130]. *J. L. Doob.*

Linnik, Yu. V. On nonstationary Markov chains. Doklady Akad. Nauk SSSR (N.S.) **60**, 21–24 (1948). (Russian)

Suppose that for every n the random variables $x_{1,n}, x_{2,n}, \cdots, x_{n,n}$ form a Markov chain. The variable $x_{j,n}$ can assume $k_{j,n}$ different values $a_{1,n}^{(j)}, a_{2,n}^{(j)}, \cdots$. It is supposed that the $a_{i,n}^{(j)}$ are uniformly bounded and that $2\leqq k_{j,n}\leqq$ constant. For fixed j, n, let $\xi_{j,n}$ be the arithmetic mean of $a_{i,n}^{(j)}$ and $\sigma_{j,n}^2$ the mean square deviation of $a_{i,n}^{(j)}$ from $\xi_{j,n}$. It is supposed that $\sigma_{j,n}$ is bounded away from zero. Suppose finally that the transition probabilities $p_{i,\nu,n}^{(j)}$ satisfy the inequality (*) $p_{i,\nu,n}^{(j)}\geqq n^{-\frac{1}{2}+\epsilon}$ for some $\epsilon>0$. Then $S_n=x_{1,n}+\cdots+x_{k_{j,n},n}$ satisfies the central limit theorem. On the other hand, this is not necessarily the case if the exponent in (*) is replaced by $\frac{1}{2}$. The proof is accomplished by an adaptation of the method of subsequences used for independent variables. The paper continues recent work of Sapogov [same Doklady **58**, 193–196, 1905–1908 (1947); these Rev. **9**, 293, 361].

W. Feller (Ithaca, N. Y.).

Kampé de Fériet, Joseph. Analyse harmonique des fonctions aléatoires stationnaires d'ordre 2 définies sur un groupe abélien localement compact. C. R. Acad. Sci. Paris **226**, 868–870 (1948).

If Ω is an abstract space on which a probability measure is defined, if $f(\omega)$ $(\omega\epsilon\Omega)$ is in L_2 and if $\{U_\alpha f\}$ is an Abelian group of unitary operators acting on L_2, then $X(\alpha)=U_\alpha f$ defines a stochastic process with parameter α. The harmonic analysis of $X(\alpha)$ then becomes that of unitary groups as developed by Stone (for the group of translations of the infinite line), implying the usual harmonic analysis of stationary processes, and generalized to abstract groups by Neumark [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 237–244 (1943); these Rev. **5**, 272], Ambrose [Duke Math. J. **11**, 589–595 (1944); these Rev. **6**, 131], and Godement [C. R. Acad. Sci. Paris **218**, 901–903 (1944); these Rev. **7**, 307]. *J. L. Doob* (Urbana, Ill.).

Frenkiel, F. N. On third-order correlation and vorticity in isotropic turbulence. Quart. Appl. Math. **6**, 86–90 (1948).

The author solves the usual Kármán-Howarth differential equation for the double and triple correlation functions by making various approximations and similarity assumptions, thereby deriving results on the decay of isotropic turbulence. For example, he finds that the decay of vorticity is proportional to $(\bar{\omega}^2)^{24/17}$. This differs from the results obtained, using different assumptions, by Batchelor and Townsend [Proc. Roy. Soc. London. Ser. A. **190**, 534–550 (1947)].

J. L. Doob (Urbana, Ill.).

Vescan, T. Introduction à l'étude statistique des équations du mouvement des fluides. Bull. Sci. École Polytech. Timişoara **9**, 287–308 (1940).

The first part of the paper is expository, devoted to the well-known Kolmogoroff partial differential equations for the transition probability of a Markoff process for a system with n degrees of freedom. In the second part, the author applies these equations, putting, for the probability distri-

bution of the process, the density ρ, the momentum $\rho\mathbf{V}$ or the energy $\rho c_v T$ of a "fluid"; he gets three equations which he calls "les équations de transfert de l'hydrodynamique." But what is meant here by a "fluid" seems not quite clear to the reviewer. *J. Kampé de Fériet* (Lille).

Mathematical Biology

Stein, Anna. A certain class of binocularly equivalent configurations. J. Opt. Soc. Amer. **37**, 944–962 (1947).

Objects in a plane passing through the two eyes (R and L) are viewed binocularly. Any such object P defines two angles $\alpha=PRL$, $\beta=PLR$, and two adjacent points P, Q define infinitesimal increments $d\alpha, d\beta$. Two elements defining respectively $(d\alpha, d\beta)$ and $(d\alpha', d\beta')$ are said to be equivalent if $d\alpha=d\alpha'$ and $d\beta=d\beta'$. This definition of equivalence is presumed to be physiologically significant. Two curves are said to be equivalent if they can be divided into corresponding equivalent elements; corresponding points are said to be coordinated. The author shows that to any given curve there corresponds a two-parameter family of equivalent curves. If the given curve is a straight line parallel to the median plane which bisects RL at right angles, the equivalent curves are conics. This family contains a one-parameter family of degenerate curves (straight lines). Thus there is a one-parameter family of straight lines equivalent to any parallel to the median plane, and this family is shown to envelope an ellipse having R and L for foci. Indeed all straight lines tangent to any ellipse having R and L for foci are equivalent; coordinated points on equivalent straight lines represent projectively corresponding points on tangents to a conic, so that the cross ratio of any four coordinated points is invariant. An experiment is suggested to test the predictions of the theory. *J. L. Synge* (Dublin).

Shimbel, Alfonso, and Rapoport, Anatol. A statistical approach to the theory of the central nervous system. Bull. Math. Biophys. **10**, 41–55 (1948).

Starting with the notions of McCulloch and Pitts [Bull. Math. Biophys. **5**, 115–133 (1943); these Rev. **6**, 12], the authors define a bulb group as the set of those end feet or bulbs on a cell body which originate in the same neuron. If there are n bulbs altogether terminating on a given cell body, and i_s represents the number of bulb groups consisting of s bulbs, then $\sum s i_s = n$ and the set of integers $i_1, i_2, \cdots, i_n$ characterizes the "synaptic type" of the neuron. Sets of n nonnegative integers such as these can be lexicographically ordered and the ordinal number p, together with the number n, designates unequivocally the synaptic type. With the threshold h the connection is fully characterized.

The statistical approach is now introduced as follows. The probability that at the point Q a neuron has synaptic type (n, p) is $S(n, p, Q)$; the probability that it has threshold h is $T(h, Q)$; the probability that a bulb group at Q originates in a region (X, dX) is $O(Q, X)dX$. Finally the probability that a neuron at X fires at time t is $f_t(X)$, so that the probability that a bulb group at Q originates at a neuron firing at $t=0$ is $I_0(Q)=\int O(Q, X) f_0(X) dX$. Hence one can express the probability that a neuron at Q shall fire at $t=1$ by means of certain polynomials $P_{n,h}$ in I_0:

$$f_1(Q)=\sum\sum S(n, p, Q)\sum T(h, Q) P_{n,h}[I_0(Q)],$$

and the other f_t are given by an iteration. These f_t are obtained explicitly in a number of special cases, interest centering particularly in the possibility that $f_{t+1} \geqq f_t$.

A. S. Householder (Oak Ridge, Tenn.).

TOPOLOGY

Bing, R. H. Extending a metric. Duke Math. J. **14**, 511–519 (1947).

L'auteur démontre un théorème de théorie des ensembles dit théorème de couverture et il l'applique à obtenir trois théorèmes de métrisation d'un espace topologique. Une famille G d'ensembles de points est dite cohérente si, pour chaque famille $G' \subset G$ telle que $G' \neq G$, il existe un ensemble de G' et un ensemble de $G-G'$ qui sont non-disjoints (c'est-à-dire qui ont un point commun).

Supposons que r est un entier positif et que $H_1, H_2, \cdots$ sont des familles d'ensembles telles que chaque ensemble de deux points qui est contenu dans la réunion d'une famille cohérente de r au plus ensembles de H_{i+1}, est contenu dans un ensemble de H_i. Si P et Q sont deux points dont l'ensemble n'est contenu dans aucun ensemble de H_i mais est contenu dans la réunion d'une famille cohérente d'ensembles $h_1, \cdots, h_n$ appartenant respectivement à $H_{a(1)}, \cdots, H_{a(n)}$, alors on a: $2(1/r^{a(1)}+\cdots+1/r^{a(n)})>1/r^i$.

Dans la suite le terme d'espace topologique pourra être pris au sens très général de Fréchet [Les Espaces Abstraits, Paris, 1928, pp. 167–168]. Le mot métrique signifiera distance de Fréchet. Une métrique $d(x, y)$ est dite presque convexe si, pour chaque point P, chaque point Q et chaque nombre positif ϵ, il existe un point R tel que $|d(P, R)-d(P, Q)/2|<\epsilon$ et $|d(R, Q)-d(P, Q)/2|<\epsilon$. L'auteur démontre les théorèmes suivants.

Un espace topologique est métrisable toutes les fois qu'il existe une suite $H_1, H_2, \cdots$ telle que les conditions (a), (b) et (c) suivantes soient vérifiées. (a) Pour chaque entier positif i, H_i est une famille d'ensembles dont la réunion coincide avec l'espace. (b) Un point P est point d'accumulation d'un ensemble M si et seulement si, pour chaque entier positif n, il existe un ensemble de H_n contenant P et non-disjoint de $M-(P)$. (c) Chaque ensemble de deux points qui est contenu dans la réunion de deux ensembles non-disjoints appartenant à H_{i+1}, est contenu dans un ensemble de H_i. Un espace topologique connexe est métrisable au moyen d'une métrique presque convexe si et seulement s'il existe une suite $H_1, H_2, \cdots$ vérifiant les conditions (a), (b), (c) et la condition (d) suivante. (d) Chaque ensemble de deux points qui est contenu dans un ensemble de H_i, est contenu dans la réunion de deux ensembles non-disjoints appartenant à H_{i+1}. Si K est un sous-ensemble fermé d'un espace métrisable S et si $d(x, y; K)$ est une métrique sur K, alors S est métrisable au moyen d'une métrique conservant $d(x, y; K)$, c'est-à-dire coincidant avec $d(x, y; K)$ sur K. *A. Appert* (Saumur).

Katětov, Miroslav. A note on semiregular and nearly regular spaces. Časopis Pěst. Mat. Fys. **72**, 97–99 (1947). (English. Czech summary)

A Hausdorff space X is hereditarily semi-regular if every subspace is semi-regular. A subset Y is regularly imbedded in X if the sets $(AY)^{-1}$ form a basis in X; X is nearly

regular if every dense Y is regularly imbedded; X is hereditarily nearly regular if every Y is nearly regular. The implications regular→hereditarily semi-regular→hereditarily nearly regular are established and shown to be irreversible. It is also shown that a hereditarily semi-regular X is regular at any point with countable basis. *R. Arens.*

Katětov, Miroslav. On the equivalence of certain types of extension of topological spaces. Časopis Pěst. Mat. Fys. **72**, 101–106 (1947). (English. Czech summary)

There are various ways in which topological spaces can be extended to compact or absolute Hausdorff spaces. The author gives necessary and sufficient conditions for the equivalence of certain pairs of the extensions considered by Čech [Ann. of Math. (2) **38**, 823–844 (1937)], Wallman [Ann. of Math. (2) **39**, 112–126 (1938)], Alexandroff [Rec. Math. [Mat. Sbornik] N.S. **5(47)**, 403–423 (1939); these Rev. **1**, 318] and the author himself [Časopis Pěst. Mat. Fys. **72**, 17–32 (1947); these Rev. **9**, 153]. *P. A. Smith.*

Sierpinski, Waclaw. Sur les espaces (V) séparables universels. Rend. Accad. Sci. Fis. Mat. Napoli (4) **10**, 357–358 (1940).

If E is a space with a denumerable set $[F(p)]$ of neighborhoods subject only to the condition that $p \varepsilon F(p)$, then E is called a separable (V)-space [M. Fréchet, Les Espaces Abstraits, Gauthier-Villars, Paris, 1928]. Two such spaces E_1, E_2 are called congruent if there is a one-to-one mapping ϕ between E_1 and E_2 such that ϕ maps each $F_1(p_1) \subset E$ onto some $F_2(\phi(p_1)) \subset E_2$ while ϕ^{-1} maps each $F_2(p_2) \subset E_2$ onto some $F_1(\phi^{-1}(p_1)) \subset E_1$. A separable (V)-space E of power m is called universal of power m if, for each such space of power m, E contains a subset E_1 with neighborhoods $E_1 \cdot F(p)$, $p \subset E_1$, congruent to it. The author gives an example, for each $m \geqq 2^{\aleph_0}$, of such a universal (V)-space of power m. The existence of such a space for $m = \aleph_0$ is asserted but the construction is omitted because of its difficulty, reference being made to a related problem [W. Sierpiński, Fund. Math. **26**, 327–333 (1936)]. *L. W. Cohen.*

Smirnov, Yu. On a class of topological spaces. Doklady Akad. Nauk SSSR (N.S.) **59**, 1253–1256 (1948). (Russian)

Alexandroff and Urysohn [Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1, **14**, no. 1 (1929)] called a topological space R (a, b)-compact if it satisfies the condition denoted in this paper by A: every subset M of R, of regular cardinal power m, $a \leqq m \leqq b$, has a point of "full accumulation"; they defined other properties, now well-known, equivalent to this one. The author generalizes these concepts by replacing the condition of regularity by a-regularity defined as follows: a cardinal m is a-regular if m cannot be represented as the sum of fewer than a cardinals each of them smaller than m. In some theorems the upper bound b is immaterial; this is indicated in definitions by the device of substituting for b the symbol ∞. There is introduced, among a number of other notions, the concept of an a-centered system. This is a system of sets such that every subcollection containing fewer than a sets has a nonvacuous intersection. The author states a variety of theorems, generally without proof. The first of these is concerned with the equivalence relations which hold between the space properties introduced in this paper; another derives a sufficient condition that a regular space be normal; there is a theorem on local (a, b)-compactness and several theorems dealing with the notion of hereditarily (a, b)-compact spaces. In some of these b is particularized, e.g., $b = a$, or $b = \infty$. *L. Zippin.*

Inagaki, Takeshi. Sur les espaces à structure uniforme. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **10**, 179–256 (1943).

This paper contains various formulations of the concept of topological uniformity and detailed generalizations of several portions of real analysis. The author calls a space S quantitative if the neighborhoods $V_\alpha(p)$ are the values of a set-function of $\alpha \varepsilon A$, an arbitrary set, and $p \varepsilon S$. A quantitative space is called a uniform structure if certain relations are satisfied by this function. Among the relations proposed are: for $p, q \varepsilon S$ and $\alpha, \beta = \beta(p, \alpha) \varepsilon A$, (A) $V_\beta(p) V_\beta(q) \neq 0$ implies $V_\beta(q) \subset V_\alpha(p)$; (B) $V_\beta(p) V_\beta(q) \neq 0$ implies $q \varepsilon V_\alpha(p)$; (C) $q \varepsilon V_\beta(p)$ implies $V_\beta(q) \subset V_\alpha(p)$; (D) $p \varepsilon V_\beta(q)$ implies $V_\beta(q) \subset V_\alpha(p)$; (E) $p \varepsilon V_\beta(q)$ implies $q \varepsilon V_\alpha(p)$; and those resulting from interchanging p and q in the above. There are uniform separation axioms such as: (T_0^*) for $p, q \varepsilon S$ there is $\alpha = \alpha(p, q) \varepsilon A$ such that no $V_\alpha(r)$ contains both p and q.

Relations among the axioms and to the work of A. Weil and L. W. Cohen are pointed out. Properties of continuous functions on uniform structures are studied. In response to a question of Fréchet [Les Espaces Abstraits, Gauthier-Villars, Paris, 1928] it is shown that in a quasi-accessible space satisfying the first denumerability axiom and in which every condensed set is locally hyper-condensed in itself, (D) plays a role in establishing the coincidence of the classes of subsets which are: (1) perfectly separable, (2) hyper-condensed in themselves, (3) possessed of the Lindelöf property, (4) condensed in themselves, (5) condensed. There follow theorems on compactness, category, connectivity and the extension of continuous functions from a subset to the space. Borel sets are studied following the ideas of D. Montgomery [Fund. Math. **25**, 527–533 (1935)] and C. Kuratowski [Fund. Math. **25**, 534–545 (1935)]. Typical results are as follows. In a T_1-space satisfying the first denumerability axiom monotonically and (D) a developable set E is simultaneously a G_δ and an F_σ [Montgomery]. If $y = f(x)$ is a Baire function of the first class mapping X which satisfies the first denumerability axiom monotonically and (D) into Y which is also perfectly separable, then the set of points of discontinuity of f is of the first category [Kuratowski]. *L. W. Cohen* (Flushing, N. Y.).

Trevisan, G. Sui campi adiacenti ad una traiettoria di una traslazione piana generalizzata. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 199–203 (1947).

This paper contains a new proof of a theorem of Brouwer concerning the complementary domain of a simple trajectory of a topological fixed-point-free transformation of the plane. *P. A. Smith* (New York, N. Y.).

Scorza Dragoni, Giuseppe. Un teorema fondamentale sulle traslazioni piane generalizzate e sua maggiore determinazione. Rend. Sem. Mat. Univ. Padova **16**, 86–158 (1947).

Let t be a fixed-point-free topological automorphism of the plane. The author establishes a number of refinements of his earlier theorems about t [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 697–704, 1163–1166 (1946); these Rev. **8**, 285, 525]. *P. A. Smith.*

Scorza Dragoni, Giuseppe. A proposito di una costruzione fondamentale per lo studio delle traslazioni piane generalizzate. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 353–365 (1947).

The author examines certain topological implications of a construction due to Kerékjártó [Acta Litt. Sci. Szeged **4**,

86–102 (1928)] and the author [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 4, 159–212 (1933)] in connection with topological plane transformations without fixed points.

P. A. Smith (New York, N. Y.).

Lihtenbaum, L. M. On certain coverings of the two-dimensional and three-dimensional toruses. Mat. Sbornik N.S. 22(64), 179–190 (1948). (Russian)

Let R be a three-dimensional torus and let $a_1, \cdots, a_k$ be closed subsets of R which in their totality cover R. Let there be defined certain mutually exclusive collections A_j whose elements are the sets a_i and let k_j denote the number of these latter sets which belong to A_j, $j=1, 2, \cdots, q$; $k=\sum k_j$. Let A_j^* denote the point-set belonging to A_j. Suppose now that the sets a_i and the sets A_j^* satisfy the following conditions: (γ) the intersection of every pair of the a_i is homeomorphic to a plane simple polygonal surface or is vacuous, the intersection of a triple of these sets is an arc or is vacuous, of four is a point or vacuous, and the intersection of every five is vacuous; (δ) the point set $A_j^* \cdot A_{j+1}^*$, $1 \leqq j \leqq q-1$, is homeomorphic to a two-dimensional torus and the intersections of the sets a_i with these toruses constitute a system with properties analogous to (γ), above. This note proves that a covering of the three-dimensional torus, delimited as in the preceding paragraph, contains some set a_i which intersects at least twelve other sets of the covering. This theorem is expressed in this form: a covering of the specified kind contains a set which has at least twelve neighbors distinct from itself.

An introductory first section establishes a numerical inequality which relates to some of the counting operations that have to be performed in the proof. The section appears to deal with a kind of finite, discrete, "neighbor"-space, but this space is adequately represented by the system of sets a_i (of the covering) where the undefined concept "is neighbor to" may be interpreted simply as "has nonvacuous intersection with."

L. Zippin (Flushing, N. Y.).

Papakyriakopoulos, Ch. A new proof for the invariance of the homology groups of a complex. Bull. Soc. Math. Grèce 22 (1943), 1–154 (1946). (Greek)

An article of expository nature in which proofs for the invariance of (a) homology groups under subdivision, (b) pseudomanifolds, (c) homogeneity are given, only for two-dimensional complexes. The author maintains that all his proofs can be extended to finite-dimensional complexes.

The following remarks are in order. (1) The idea of the proof of (a) follows that indicated in Hurewicz and Wallman's Dimension Theory [Princeton University Press, 1941, pp. 122–123; these Rev. 3, 312]. This book was, however, presumably not available to the author. (2) There are several errata. For example, the lemma of § 26, page 23, as stated, cannot be correct. Also, because of the cumbersome notation used, the proof of (a) given by the author contains a serious gap. In fact, denoting by $C_n(K)$ the chain group of a complex K, and by K' a subdivision of K, the author does not distinguish between the usual chain mappings $f: C_n(K') \to C_n(K)$ and $g: C_n(K) \to C_n(K')$. He uses the symbolism $\varphi' = \varphi$ to denote either $f(\varphi') = \varphi$ or $g(\varphi) = \varphi'$, and this leads him, in part B, page 151, to hastily assume, incorrectly, that gf is the identity transformation of $C_n(K')$.

J. Dugundji (Los Angeles, Calif.).

Bokštein, M. On the Alexander-Kolmogorov duality theorem. Doklady Akad. Nauk SSSR (N.S.) 59, 631–633 (1948). (Russian)

By a modification of a proof due to P. Alexandroff [Trans. Amer. Math. Soc. 49, 41–105 (1941); these Rev. 2, 323] of the Alexander-Kolmogorov duality theorem, the author eliminates the condition that the locally bicompact Hausdorff space R of that theorem be a normal space.

L. Zippin (Flushing, N. Y.).

Spanier, Edwin H. Cohomology theory for general spaces. Ann. of Math. (2) 49, 407–427 (1948).

L'auteur étudie une nouvelle définition des groupes de cohomologie valable pour des espaces topologiques généraux. Voici cette définition, inspirée de J. W. Alexander [Proc. Nat. Acad. Sci. U. S. A. 21, 509–511, 511–512 (1935)] et proposée, paraît-il, par A. D. Wallace: G étant un groupe abélien discret donné une fois pour toutes, le groupe $\Phi^p(X)$ des cochaînes de dimension p d'un espace X est le groupe des fonctions $f(x_0, \cdots, x_p)$ de $p+1$ arguments $x_i \varepsilon X$ à valeurs dans G; on définit un opérateur "cobord" δ de $\Phi^p(X)$ dans $\Phi^{p+1}(X)$:

$$\delta f(x_0, \cdots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i f(x_0, \cdots, \hat{x}_i, \cdots, x_{p+1});$$

les cocycles sont les f telles que $\delta f = 0$, les cobords les f de la forme δg; tout cobord est un cocycle, à cause de $\delta\delta = 0$. La topologie de X permet de définir, pour chaque entier $p \geqq 0$, le sous-groupe $\Phi_0^p(X)$ des cochaînes "localement nulles," c'est-à-dire nulles dans un voisinage de la diagonale de X^{p+1}; en identifiant deux cochaînes dont la différence est localement nulle, on a, dans la somme directe des Φ^p/Φ_0^p, un opérateur $\bar{\delta}$ défini à partir de δ par passage au quotient. On obtient ainsi les cocycles et cobords des groupes quotients, d'où le "groupe de cohomologie de l'espace X."

Les groupes ainsi définis coïncident avec les groupes de cohomologie de Čech si X est compact, mais ils sont plus maniables car on peut travailler avec l'opérateur $\bar{\delta}$. Lorsque X est l'espace d'un complexe infini, localement fini, les groupes de Wallace sont identiques aux groupes de cohomologie de la "théorie singulière." [Note du reviewer: plus généralement, si X est localement compact mais non compact, et si on modifie la définition de Čech en remplaçant les recouvrements ouverts finis par les recouvrements localement finis formés d'ouverts relativement compacts [cf. Dowker, Amer. J. Math. 69, 200–242 (1947); ces Rev. 8, 594], on retrouve les groupes de cohomologie de Wallace, au moins si X est paracompact au sens de Dieudonné.]

Les démonstrations sont basées sur la théorie axiomatique d'Eilenberg-Steenrod [Proc. Nat. Acad. Sci. U. S. A. 31, 117–120 (1945); ces Rev. 6, 279]. Pour l'appliquer, il faut définir les groupes de Wallace "relatifs" (pour un espace X et un sous-espace A); ils satisfont à tous les axiomes d'Eilenberg-Steenrod, et aussi à l'"axiome de continuité."

[Note du reviewer: en définissant le "support" d'une cochaîne f de $\Phi^p(X)$ comme le complémentaire de l'ensemble des points $x \varepsilon X$ tels que $f(x_0, \cdots, x_p) = 0$ pour tout point $(x_0, \cdots, x_p) \varepsilon X^{p+1}$ assez voisin de $(x, \cdots, x)$, on est conduit aux "groupes de cohomologie à supports compacts" qui jouent un rôle important dans le cas des espaces localement compacts.]

H. Cartan (Paris).

Dowker, C. H. An extension of Alexandroff's mapping theorem. Bull. Amer. Math. Soc. 54, 386–391 (1948).

L'auteur étudie les applications continues d'un espace topologique (qui n'est pas supposé vérifier aucun axiome de

séparation) dans le nerf d'un recouvrement ouvert quelconque de cet espace (ce nerf étant muni de l'une ou l'autre de deux topologies, que l'auteur appelle respectivement "naturelle" et "géométrique"); une telle application est dite canonique si l'image réciproque de l'étoile d'un sommet quelconque u_α du nerf d'un recouvrement $\mathfrak{U}$ est contenue dans l'ensemble U_α du recouvrement $\mathfrak{U}$ qui correspond à u_α. Généralisant un théorème d'Alexandroff, l'auteur montre que pour qu'il existe une application canonique d'un espace R dans le nerf d'un recouvrement arbitraire de R, il faut et il suffit que R soit paracompact et normal. Il précise alors (suivant toujours Alexandroff) les propriétés de certaines de ces applications canoniques en relation avec la notion de dimension. *J. Dieudonné* (Nancy).

Youngs, J. W. T. Homeomorphic approximations to monotone mappings. Duke Math. J. **15**, 87–94 (1948).

The following theorem is proved. If m is a monotone mapping defined over a compact 2-manifold X, with or without boundary, such that $m(X)=X$, then given $\epsilon>0$ there exists a homeomorphism $h_\epsilon(X)=(X)$ such that for every $x\varepsilon X$ we have $\rho[m(x), h_\epsilon(x)]<\epsilon$. This extends results of Radó [Trans. Amer. Math. Soc. **58**, 420–454 (1945); these Rev. **7**, 282] and Youngs [Proc. Nat. Acad. Sci. U. S. A. **32**, 328–330 (1946); these Rev. **8**, 259], who proved the theorem for X a closed 2-cell and a 2-sphere, respectively. The proof makes use of the results of Roberts and Steenrod [Ann. of Math. (2) **39**, 851–862 (1938)] and of the fundamental theorem of Schoenflies to the effect that a homeomorphism between two plane simple closed curves can be extended to their interiors. *J. H. Roberts.*

GEOMETRY

***Todd, J. A. Projective and Analytical Geometry.** Pitman Publishing Corporation, New York, 1946. x+289 pp. $4.50.

This is an accurate and readable account of complex projective geometry from the analytic standpoint. The number of dimensions is at first arbitrary but later specialized to 2 or 3. The author begins with an $(n+1)$-dimensional vector space, calling the class of scalar multiples of a given vector a ray. He sets up the projective n-space by calling these rays points, thus avoiding the customary discussion of "points at infinity." Other topics treated in the first chapter are the n-dimensional propositions of incidence, duality and the coordinates of a line in 3-space. Chapter II deals with one-dimensional projectivities, including an unusually thorough account of the invariants and covariants of a binary form, in preparation for the theory of correspondences on a conic in chapter III. In chapter IV the twisted cubic is defined by means of three axial pencils of planes, great care being taken to avoid degeneracy. This leads naturally to a discussion of null systems, linear complexes and Klein's representation of lines in 3-space by points in 5-space. Chapter V includes an exhaustive classification of collineations, using the Segre characteristic. This is related to the classification of linear systems of quadrics in chapter VI. The final and most original chapter is probably the best account ever written of the invariants and covariants of quadratic forms, and of their geometrical interpretation in terms of conics and quadrics. It ends with a discussion of self-polar pentads and hexads: sets of five or six points such that the pole of the plane determined by three of the points lies on the line or plane determined by the remaining two or three. A special feature of the book is the vast number of exercises, many of which are from Cambridge University examination papers. There are also 24 figures and a good index. *H. S. M. Coxeter.*

Edge, W. L. The Klein group in three dimensions. Acta Math. **79**, 153–223 (1947).

The Klein group is the simple group of order 168, exhibited by Klein as a group of ternary linear substitutions for which a plane quartic k is invariant. An isomorphic group of collineations in three-dimensional space is studied by the author and the geometry of the figure is that of a net of quadrics invariant for the group. His starting points are the original publications of Klein, a note of Baker [Proc. Cambridge Philos. Soc. **31**, 468–481 (1935)] and his own investigations on nets of quadric surfaces [Proc. London Math. Soc. (2) **43**, 302–315 (1937); **44**, 466–480 (1938); **47**, 123–141 (1941); J. London Math. Soc. **12**, 276–280 (1937); cf. these Rev. **3**, 15]. The left side of the equation of k may be obtained as a symmetrical four-rowed determinant, the elements being linear forms in the coordinates (ξ, η, ζ). This determinant can be regarded as the discriminant of a quaternary quadratic form; this form, when equated to zero, is a quadric which, as ξ, η, ζ vary, varies in a net of quadrics. The locus of the vertices of the cones of the net is its Jacobian, a sextic twisted curve K, which is hereby put in one-to-one correspondence with k. Six coplanar points of K correspond to six points of k which are the points of contact of k with a contact cubic. There are 36 systems of contact cubics; one of these is so associated with the planes of space and the group of 168 ternary substitutions is associated with a group of collineations in space for which the net is invariant. The net has eight base points and eight base planes; they are permuted by the collineations of the group. There are 24 points c on K where the osculating plane has five-point contact, and the sixth intersection with K is a point c itself. The number of involutions of the group G_{168} is twenty-one; they are all biaxial; the forty-two axes consist of fifty-six triads of lines, the lines of each triad being both concurrent and coplanar; the axes are principal chords of K. There is one and only one quartic surface invariant for G; it intersects K in the 24 points c. There is one and only one sextic surface invariant for G. The invariant octavic surfaces are investigated; when they do not contain K they touch K at each of the 24 points c. The author studies the covariant line complexes of the group and makes remarks on the covariant cubic and quartic complexes; this material is obviously relevant to a group of 168 substitutions on six variables. *O. Bottema* (Delft).

Bydžovský, B., and Knichal, V. On the simultaneous invariant Φ of two quadrics. Rozpravy II. Třídy České Akad. **50**, no. 21, 10 pp. (1940). (Czech)

Two quadrics in ordinary space have three invariants: θ_1, ϕ, θ_2. The condition $\phi=0$ is necessary and sufficient for the existence of a polar tetrahedron of one of the quadrics whose edges are tangent to the other. For this well-known property the authors give an elaborate proof by considering the different Segre types. *H. A. Lauwerier.*

Bunch, W. H. The quadrilaterals of Pascal's hexagram. Amer. Math. Monthly **55**, 210–217 (1948).

The purpose of this paper is to show that the 60 Pascal lines form 15 distinct quadrilaterals whose vertices are Pascal points and whose 45 diagonals are the Brianchon lines. The relationship of these quadrilaterals to the results obtained by others is discussed. *H. A. Lauwerier.*

Emch, Arnold. Eigenschaften von Flächen zweiter Ordnung, hergeleitet mit Hilfe stereographischer Projektion. Elemente der Math. **3**, 64–65 (1948).

Bogdan, C. P. Sur les surfaces lieu de coniques. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **2**, 121–124 (1947).

Sypták, M. Les spirales d'ordre m dans l'espace euclidien au nombre quelconque de dimensions. Časopis Pěst. Mat. Fys. **72**, 107–127 (1947). (Czech. French summary)

On étudie certaines courbes de l'espace euclidien R_p qui sont exprimées dans le système de coordonnées rectangulaires par les équations $x_{2i-1}=r_i k\cdot s^m\cdot\cos l_i s$, $x_{2i}=r_i k\cdot s^m\cdot\sin l_i s$, respectivement $x_{2n+1}=Ck\cdot s^m$, $i=1,2,\cdots,n$, où $r_1^2+\cdots+r_n^2=1$, respectivement $r_1^2+\cdots+r_n^2+C^2=1$, et $r_1^2l_1^2+\cdots+r_n^2l_n^2=1$, et appellées les spirales d'ordre m. Ces courbes sont bien connues dans le plan R_2 où elles peuvent être exprimées dans le système polaire ρ, ω par l'équation $\rho=k\cdot\omega^m$. La plupart des propriétés de ces spirales planes sont conservées dans l'espace au nombre quelconque de dimensions. Dans l'appendice sont indiquées quelques propriétés de la cochléoïde donnée dans R_{2n} par les équations $x_{2i-1}=s^{-1}R_i(1-\cos l_i s)$, $x_{2i}=s^{-1}R_i\sin l_i s$. *From the author's summary.*

Schuster, Jan. Contributions à la géométrie du triangle et du tétraédre. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd **1947**, no. 4, 7 pp. (1948). (Czech. French summary)

In a triangle it is possible to derive, by means of a simple geometric construction, from a point P with normal coordinates (α, β, γ) a point $P(a\alpha, b\beta, c\gamma)$. By repeating or inverting this process points $P_n(a^n\alpha, b^n\beta, c^n\gamma)$ and $P_{-n}(a^{-n}\alpha, b^{-n}\beta, c^{-n}\gamma)$ are found. Permutation of the coordinates of P gives five other points, which together with P lie on a conic. These results can be partially extended for a tetrahedron. *H. A. Lauwerier* (Amsterdam).

Gambelli, Lucio. Le concoidi nella risoluzione di problemi sul triangolo. Boll. Un. Mat. Ital. (3) **2**, 239–244 (1947).

Court, N. A. Skewly Cevian tetrahedrons. Duke Math. J. **15**, 49–54 (1948).

Two tetrahedra are defined as skewly Cevian if the lines connecting corresponding vertices are elements of a regulus. The author establishes some theorems about tetrahedra so related. *R. A. Johnson* (Brooklyn, N. Y.).

Court, N. A. Notes on cospherical points. Amer. Math. Monthly **55**, 218–221 (1948).

Given n points on a sphere, we may associate with any one of them the centroid of the remaining $n-1$ points. It is shown that the n median points are homothetic to the given points and hence lie on a sphere, the medial sphere. Properties of the external sphere of antisimilitude of the two spheres are developed; for instance, that spheres on the medians as diameters are orthogonal to the last-named spheres. *R. A. Johnson* (Brooklyn, N. Y.).

Maxwell, E. A. Some properties of the nine-points circle. Math. Gaz. **31**, 266–269 (1947).

This paper uses the traditional "abridged notation" wherein a single symbol represents the equation of a line, circle or other locus. This method facilitates operations of linear dependence, and the author deduces numerous properties of the nine-point circle and other lines and circles related to a triangle. His epilogue is worth quoting: "I ought to add that I cannot believe that much of the above work is new, though I am unable to find it in what are to me the standard textbooks. Anyway, it does not seem to have been done recently, and a new generation may like to be reminded of the figure. Obviously there are numerous points where fresh investigations are suggested, and that is my chief excuse for writing." *R. A. Johnson* (Brooklyn, N. Y.).

Pompeiu, D. Un théorème de géométrie. Acad. Roum. Bull. Sect. Sci. **24**, 223–226 (1943).

Join the midpoints of the sides of a closed planar polygon having congruent sides with any point of the plane of the polygon. The author shows by means of an identity for complex numbers that from the segments thus obtained a second closed polygon can be formed. *L. Fejes Tóth.*

Zacharias, Max. Ein geometrischer Satz. Acad. Roum. Bull. Sect. Sci. **25**, 247–248 (1943).

Let $a_1, a_2, \cdots, a_{2n}$ be $2n$ vectors and put $b_i=\frac{1}{2}(a_i+a_{i+1})$ $(i=1, 2, \cdots, 2n;\ a_{2n+1}\equiv a_1)$. The author makes the remark that the trivial equality $b_1-b_2-\cdots-b_{2n}=0$ involves for an even number of the sides a result of Pompeiu [see the preceding review] without restriction to polygons having sides of the same length. *L. Fejes Tóth* (Budapest).

Roth-Desmeules, E. Noch eine Aufgabe, die mit Zirkel und Lineal nicht lösbar ist. Elemente der Math. **3**, 65–67 (1948).

The problem is the construction of a triangle, given two sides and the bisector of an angle opposite one of them.

Convex Domains, Integral Geometry

Fejes Tóth, László. An inequality concerning polyhedra. Bull. Amer. Math. Soc. **54**, 139–146 (1948).

Let V_n and v_n denote the volumes of two ellipsoids, the first containing, the second contained in, a convex polyhedron having either n vertices or n faces. Then $V_n/v_n\geqq 3^{3/2}\tan^3[n\pi/(6n-12)]$. This estimate is asymptotically exact for $n\to\infty$, and equality holds for affine regular polyhedra, but only in the cases $n=4, 6, 12$. This generalizes a previous result of the author [Mat. Fiz. Lapok **50**, 40–46 (1943); these Rev. **8**, 219], and inequalities for the density of systems of congruent segments covering a sphere due to Kershner [Amer. J. Math. **61**, 665–671 (1939); these Rev. **1**, 8] and the author [Jber. Deutsch. Math. Verein. **53**, 66–68 (1943); these Rev. **8**, 167] are corollaries. *W. Fenchel* (Copenhagen).

Fejes Tóth, László. Approximation by polygons and polyhedra. Bull. Amer. Math. Soc. **54**, 431–438 (1948).

Let J be a Jordan curve (plane or skew) the curvature $G(s)$ of which is a continuous function of the length of arc s, P_n an arbitrary inscribed n-gon, and $\eta(P_n, J)$ the deviation of the two curves. The author defines the approximability

a of J by $1/a=\lim_{n\to\infty}$ g.l.b. $n^2\eta(P_n, J)$ and proves that $1/a=\frac{1}{4}(\int_0^l|G(s)|^2ds)^2$. From this it follows that among all curves with given length l the circle has the smallest approximability. Similar results concerning the approximation of convex curves by circular arc polygons and of convex surfaces by inscribed polyhedra are formulated and the proofs are sketched. *W. Fenchel* (Copenhagen).

Hadwiger, H. Die isoperimetrische Ungleichung im Raum. Elemente der Math. **3**, 25–38 (1948).

A rather elementary proof of the isoperimetric inequality for convex bodies of ordinary space, the main tools being Steiner's symmetry transformation and outer and inner parallel bodies. In order to show that equality holds only for the sphere a sharpened inequality containing the radius of the inscribed sphere is proved. *W. Fenchel.*

Hadwiger, H. On a symbolic topological formula. Gaz. Mat., Lisboa **9**, no. 35, 6–9 (1948). (Portuguese)

Translated from Elemente der Math. **2**, 35–41 (1947); these Rev. **8**, 484.

Kinder, W. Eine einfache Beziehung zwischen der Oberfläche und dem Profil eines Körpers. Optik **2**, 396–398 (1947).

Cauchy's formula for the surface area of a convex body is proved. The author seems not to be aware that it is well known to mathematicians. *W. Fenchel* (Copenhagen).

Santaló, L. A. On the convex bodies of constant width in E_n. Portugaliae Math. **5**, 195–201 (1946). (Spanish)

Let C be a convex body in E^n. The volume of the exterior parallel body to C at distance h is $\sum\binom{n}{r}h^rW_r$, where W_r are the mixed volumes of C and the unit sphere. If C has constant width a then the following independent relations hold for the W_r. If $n=2m$ then

$$2W_{2r-1}+\sum_{i=1}^{n-2r+1}(-1)^i\binom{n-2r+1}{i}a^iW_{2r-1+i}=0,\qquad r=1,\cdots,m.$$

If $n=2m+1$ then

$$2W_{2r}+\sum_{i=1}^{n-2r}(-1)^i\binom{n-2r}{i}a^iW_{2r+i}=0,\qquad r=0,1,\cdots,m.$$

These relations contain all the relations previously known for $n=2$ and $n=3$. *H. Busemann* (Los Angeles, Calif.).

Santaló, L. A. Some integral formulas referring to convex bodies. Revista Unión Mat. Argentina **12**, 78–87 (1946). (Spanish)

Let (θ, φ) be the point where a given line g in E^3 intersects the unit sphere with θ and φ as colatitude and longitude. In a plane normal to g introduce a rectangular coordinate system. If g intersects the plane at (x, y), then the differential $dg=\sin\theta d\theta d\varphi dxdy$ is used to measure sets of lines in E^3. For a given convex body K the set of those lines g which do not intersect K and where at least one of the two supporting planes to K through g contains more than one point of K has measure 0.

If g is a line which does not intersect K and if each of the two supporting planes π_i through g touches K in exactly one point p_i, let t_i denote the segment (and its length) from p_i normal to g and ω the angle formed by t_1 and t_2. Then $\int t_1^{-1}t_2^{-1}\sin^3\omega dg=8\pi^2$. If M is the integral over the mean curvature of K then $\int(2\omega-\sin 2\omega)(t_1^{-1}+t_2^{-1})dg=16\pi M$. If the Gauss curvature of K is everywhere different from 0 and K_i is the Gauss curvature at p_i then $\int(2\omega-\sin 2\omega)(t_1^{-1}K_1^{-1}+t_2^{-1}K_2^{-1})dg=4MF$, where F is the area of K. *H. Busemann* (Los Angeles, Calif.).

Algebraic Geometry

Severi, Francesco. Il punto di vista gruppale nei vari tipi di equivalenza sulle varietà algebriche. Comment. Math. Helv. **21**, 189–224 (1948).

The various concepts of equivalence on an algebraic variety were classified by J. A. Todd [Ann. of Math. (2) **35**, 702–704 (1934)]. The paper under review is virtually an essay on Todd's ideas. The subject-matter treated is, of course, largely the creation of the author. It is claimed that the definition given of the algebraic equivalence of varieties on an algebraic variety [§ 5] must rank as definitive.

D. Pedoe (London).

Dubreil, Paul. Variétés arithmétiquement normales et variétés de première espèce. C. R. Acad. Sci. Paris **226**, 548–550 (1948).

Let $\mathfrak{a}$ be a homogeneous ideal in the ring

$$R_n=K[x_0,\cdots,x_n],$$

where K is algebraically closed. In the representation $\mathfrak{a}=q_0\cap q_1\cap\cdots\cap q_r$, the q's being primary ideals, a q_i is "essential" if the associated prime ideal is not the ideal $(x_0,\cdots,x_n)$. Then $\mathfrak{a}$, and also the associated variety V, is said to be of the "first kind" if, for any form Φ not belonging to any essential component q_i of $\mathfrak{a}$, $\mathfrak{a}+(\Phi)$ has only essential components. If $\mathfrak{a}$ is any prime ideal in R_n the associated irreducible variety V is "arithmetically normal" if the ring $\sigma=R_n/\mathfrak{a}$ is integrally closed. Similarity in the behavior of linear systems on varieties of these two special types suggests some connection between them and the author proves that every arithmetically normal variety is of the first kind. The proof makes use of properties of quasi-equal ideals in σ, which can be done since σ is integrally closed.

R. J. Walker (Ithaca, N. Y.).

Gaeta, Federico. On the arithmetically normal curves and surfaces of S_r. Revista Mat. Hisp.-Amer. (4) **7**, 255–268 (1947). (Spanish)

Les variétés algébriques irréductibles arithmétiquement normales V de S_r ont été étudiées par Zariski [Amer. J. Math. **61**, 249–294 (1939)]. H. T. Muhly [Ann. of Math. (2) **42**, 921–925 (1941); ces Rev. **3**, 15] a caractérisé ces variétés V par la propriété que les hypersurfaces d'ordre quelconque de S_r déterminent sur V un système linéaire complet. Eu égard aux courbes régulières arithmétiquement normales, l'auteur établit que si, par une courbe arithmétiquement normale C de S_r on fait passer $r-1$ hypersurfaces se coupant, en dehors de C, suivant une courbe C' irréductible et non singulière, C' est arithmétiquement normale comme C. La démonstration de cette proposition est basée sur la propriété de Muhly ci-dessus rappelée, et utilise les formules de postulation que Severi a données pour les intersections complètes et la formule de Noether sur le genre de la courbe réductible $(C+C')$. Le théorème précédent est appliqué aux courbes de résiduel fini de S_3. L'auteur pose la question de savoir si une courbe arithmétiquement normale de S_r est de résiduel fini, et y repond

par l'affirmative dans le cas ou $r=3$. Généralisant ensuite son théorème aux hypersurfaces régulières de S_r, il établit que si F et F' sont deux surfaces irréductibles sans points multiples de S_r ($r\geqq 4$) constituant l'intersection complète (simple) de $r-2$ hypersurfaces, F' est arithmétiquement normale si F l'est. Comme conséquence de cette dernière proposition, il est montré que toute surface de résiduel fini de S_r, $r\geqq 4$, est régulière et arithmétiquement normale.

P. Vincensini (Besançon).

Gaeta, Federico. Postscript to a note on skew algebraic curves of residual one. Revista Acad. Ci. Madrid **41**, 339–350 (1947). (Spanish)

[The note in question appeared in Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 78–81 (1947); these Rev. **9**, 156.] The author proves the following theorem. Let W_d be an irreducible nonsingular algebraic variety in S_r such that (1) primals of arbitrary order l cut on W_d a complete linear system, (2) primals of some order $\rho(\geqq 0)$ cut the complete canonical system on W_d. Let V_d be a second irreducible nonsingular variety in S_r such that V_d+W_d is the complete (simple) intersection of $r-d$ primals $f_1=0, f_2=0, \cdots, f_{r-d}=0$. Then there exists a primal $f_{r-d+1}=0$ such that the $r-d+1$ forms f_i form a base for the polynomial ideal defined by V_d. If, conversely, V_d is irreducible and nonsingular, and if the polynomial ideal defined by V_d has a base consisting of $r-d+1$ forms $f_1, \cdots, f_{r-d+1}$, such that the residual intersection (simple) of $f_1, \cdots, f_{r-d}$ is irreducible and nonsingular, then W_d possesses the properties (1) and (2) above. The rest of the paper is devoted to a classification of curves in S_3 of residual one.

J. A. Todd (Cambridge, England).

Lovrecich, Gabriella. Sulle singolarità della curva Hessiana. Boll. Un. Mat. Ital. (3) **2**, 129–132 (1947).

Les conditions pour que la Hessienne H d'une courbe algébrique plane F d'ordre $n>2$ admette un point multiple d'ordre τ, $0\leqq\tau\leqq 3(n-2)$, en un point simple O de F ont été données sous forme générale par Villa [Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (2) **65**, 625–641 (1932)]. Dans l'article actuel l'auteur explicite ces conditions dans le cas où $\tau=3$, et donne des conditions suffisantes pour $\tau=4$. Pour $\tau=3$: pour qu'un point simple O d'une courbe algébrique F soit triple pour la Hessienne H de F, il faut et il suffit que O soit un point d'inflexion de 3ème espèce pour F, et qu'en outre la cubique γ de décomposition de la quartique polaire de O par rapport à F admette, comme point double, le point d'intersection de la tangente inflexionnelle en O et de la polaire harmonique de ce même point (droite autre que la tangente inflexionnelle suivant laquelle se décompose la conique polaire de O par rapport à F). La démonstration utilise un résultat antérieur de Del Pezzo, suivant lequel, pour $\tau=2$, O est nécessairement un point d'inflexion de 2ème espèce pour F (point en lequel la tangente coupe la courbe en quatre points confondus). Le choix d'un système de référence (x_1, x_2, x_3) pour lequel $O(0, 0, 1)$ est le point considéré sur F, $x_2=0$ et $x_3=0$ étant respectivement la tangente inflexionnelle et la polaire harmonique, conduit au résultat avec le maximum de simplicité. C'est par l'emploi de ce même système de référence que se trouvent établies des conditions suffisantes relatives au cas où $\tau=4$ et qui s'énoncent ainsi: un point simple O d'une courbe algébrique plane F est quadruple pour la Hessienne de F si aux conditions relatives au point triple formulées dans l'énoncé précédent on ajoute les suivantes: O est un point d'inflexion de 4ème espèce pour F (cette condition est aussi nécessaire), en outre la quartique suivant laquelle se décompose la quintique polaire de O par rapport à F admet un point triple à l'intersection de la polaire harmonique et de la tangente inflexionnelle, deux des trois tangentes en ce point triple coïncidant avec la tangente inflexionnelle.

P. Vincensini (Besançon).

Longhi, Ambrogio. Sulle cubiche razionali. Elemente der Math. **3**, 51–54 (1948).

Segre, B. Caratterizzazione geometrica degli integrali abeliani e dei loro residui. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 167–172, 172–179 (1947).

In these two notes the author discusses the zeros and poles of the differential dI associated with an Abelian integral of an algebraic curve. Among other results projective interpretations for the ratios of the residues of I are considered. *O. F. G. Schilling* (Chicago, Ill.).

Derwidué, L. Recherches sur les transformations birationnelles. Mém. Soc. Roy. Sci. Liége (4) **7**, 197–366 (1946).

Verfasser geht von einer durch Doehlemann und Castelnuovo zuerst behandelten Fragestellung aus: Die Cremonatransformationen einer Ebene, bzw. eines R_3 auf sich zu klassifizieren, die eine Kurve, bzw. Fläche von Fixpunkten besitzen. Statt dieser bald auf Schwierigkeiten führenden Frage behandelt der Verfasser die Cremonatransformationen unter einem andern, damit zusammenhängenden Gesichtspunkt und unterscheidet bei ebenen Transformationen folgende 2 Klassen: (a) solche, die ein Büschel von Kurven in sich überführen, (b) alle übrigen, und im Raume folgende 3 Klassen: (a) solche Transformationen, die eine Kongruenz von Kurven in sich überführen, (b) solche, die ein Büschel von Flächen in sich überführen, ohne zur Klasse (a) zu gehören, (c) alle übrigen. Im ersten Kapitel werden nun zunächst einige weiterhin gebrauchte Tatsachen über Büschel von Kurven und Flächen sowie über Kurvenkongruenzen im Raum entwickelt. Es wird hier z. B. der Satz entwickelt bewiesen: Gegeben ist ein Büschel von Flächen und auf jeder Fläche ein System von Kurven, die durch eine rational von den Büschelparametern abhängende Menge von Linearsystemen anderer Flächen ausgeschnitten werden; dann gibt es auch ein Linearsystem von Flächen, die gleichzeitig aus allen Flächen des Büschels die Kurven des Systems ausschneiden.

Im Kapitel II werden dann folgende Fundamentalsätze bewiesen: (a) in der Ebene: Jede Cremonatransformation der Ebene, die die ∞^1 Kurven eines Büschels invariant lässt, lässt gleichermassen auch die Kurven eines Büschels von ∞^1 rationalen oder elliptischen Kurven invariant; (b) im Raum: Diejenigen Transformationen, die die ∞^1 Kurven einer linearen Kongruenz invariant lassen, tun dies auch mit denen einer linearen Kongruenz rationaler, elliptischer oder hyperelliptischer Kurven. Solche Transformationen sind stets zyklisch; (c) im Raum: Diejenigen Transformationen, die nicht zum vorherigen Typ gehören und ein Büschel invarianter Flächen besitzen, besitzen auch ein solches Büschel invarianter Flächen, die entweder rational sind oder zu elliptischen oder hyperelliptischen Regelflächen birational verwandt sind. Mit Hilfe dieser Fundamentalsätze wird dann die Untersuchung der Cremonatransformationen mit Fixbüschel oder Fixkongruenz auf die einer bestimmten Zahl von Typen zurückgeführt. Die nächsten

Kapitel dieser breit angelegten Arbeit enthalten darüber weitgehende Einzeluntersuchungen.

Kapitel III bringt zunächst eine Klassifikation aller ebenen und räumlichen Transformationen mit einem Büschel, bzw. einer Kongruenz von rationalen Kurven. Benutzt werden hierbei die bekannten Sätze, dass ein Büschel rationaler Kurven in der Ebene stets einem Geradenbüschel und eine Kongruenz rationaler Kurven im Raum entweder einem Bündel von Geraden oder einer gewissen linearen Kongruenz von Kegelschnitten birational äquivalent ist. Hiermit ergeben sich die gesuchten birationalen Transformationen ihrem Typ nach dadurch, dass man auf jeder rationalen Kurve des Systems im Normalfall eine Involution vorschreibt und diese Involutionen zu einer Gesamttransformation der Ebene, bzw. des Raumes zusammensetzt.

Im Kapitel IV werden dieselben Transformationstypen untersucht, wobei jedoch an Stelle der rationalen Kurven elliptische oder hyperelliptische treten. Es werden alle wohlbekannten Typen von Selbstabbildungen einer elliptischen Kurve auf sich diskutiert und festgestellt, wie man mit ihrer Hilfe birationale Transformationen aufbauen kann, bei denen diese Kurven als Ganzes festbleiben. Im Falle, dass das System der fest bleibenden Kurven aus hyperelliptischen Kurven besteht, wird man dabei nur zu umkehrbaren Cremonaabbildungen, d.h. Involutionen geführt. Es ergibt sich dann folgender Satz: Notwendige und hinreichende Bedingung für die Rationalität einer räumlichen Involution ist, dass die Punktgruppen derselben sich auf die Kurven einer linearen Kongruenz verteilen und daselbst rational ausgeschnitten werden können.

Im Kapitel V wird dann die anfängliche Fragestellung wieder aufgegriffen, die Cremonatransformationen der Ebene zu klassifizieren, die eine ganze Kurve von Fixpunkten besitzen. Es war schon früher durch Godeaux gezeigt worden, dass jede beliebige birationale Transformation einer Ebene in sich durch Anwendung einer geeigneten Cremonatransformation auf ihre Ebene in eine solche mit einer ganzen Kurve von Fixpunkten übergeführt werden kann. Diese wiederum gehören zu einer von folgenden 3 Klassen: (1) Solche Transformationen, die die Strahlen eines Büschels invariant lassen, (2) solche, die die Kurven eines Halphenschen Büschels invariant lassen, (3) solche, die kein invariantes Büschel besitzen. Besonders bemerkenswert ist hierbei der Typ (3); die Fixpunktkurve besteht dann aus Komponenten vom Geschlecht 0 oder 1. Es werden in der Arbeit zum ersten Male hierher gehörige Transformationen konstruiert, die eine nicht zu einer Kurve 3. Grades Cremonaäquivalente elliptische Kurve als Fixpunktkurve besitzen. In einem besonderen Paragraphen werden einige Spezialfälle derartiger Transformationen konstruiert: solche mit einer zerfallenden Kurve 6. Grades und solche mit einer Kurve 9. Grades mit 10 dreifachen Punkten als Fixpunktkurven. Schliesslich werden auch ebene Transformationen mit einer r-fach zählenden Fixpunktkurve C betrachtet und gezeigt, dass für $r>1$ das Geschlecht von C höchstens 1 beträgt; Transformationen dieser Art sind für jedes r möglich, wenn C dabei eine beliebige Kubik ist.

Im Kapitel VI werden dann die Transformationen im einzelnen behandelt, die ein Büschel rationaler oder zu Regelflächen verwandter Flächen invariant lassen. Zunächst werden die Büschel rationaler Flächen ihrem birationalen Typ nach klassifiziert und z. B. festgestellt, dass alle Flächen derartiger Büschel solche mit elliptischen ebenen Schnitten sein müssen. Jedes Büschel dieser Art gestattet dann aber auch, wie weiter gezeigt wird, die Konstruktion zugehöriger Cremonatransformationen, die die Flächen desselben fest lassen und ausserdem noch eine Fixpunktkurve oder -Fläche besitzen. Dieselbe Frage gestellt für ein Büschel von Flächen, die auf Regelflächen abbildbar sind, fällt unter die nach Transformationen, die eine lineare Kongruenz rationaler Kurven invariant lassen. Derartige Transformationen existieren aber nur in dem Fall, wo die Kongruenz rationaler Kurven einem Geradenbüschel äquivalent ist, aber nicht in dem zweiten möglichen Fall.

Kapitel VII behandelt dann die hyperelliptischen Flächen, bei denen es bekanntermassen stets am Periodengitter leicht zu beschreibende Transformationen in sich gibt, während in Spezialfällen darüber hinaus noch weitere sog. singuläre Transformationen existieren. Es wird nun folgende Frage gestellt, jedoch entsprechend der grösseren Schwierigkeit nicht mit der gleichen Vollständigkeit wie die entsprechenden in den vorherigen Kapiteln gelöst: Wie weit lassen sich solche hyperelliptischen Flächen zu Büscheln zusammenstellen und räumliche Transformationen angeben, die jede Fläche des Büschels in sich überführen und auf ihr eine reguläre oder singuläre Transformation in sich induzieren.

Kapitel VIII schliesslich behandelt die Angabe von räumlichen Transformationen mit einer Kurve oder Fläche von Fixpunkten. Es wird darin gezeigt, dass die Fixpunktfläche eine Komponente besitzen muss, die ein Büschel rationaler oder elliptischer Kurven enthält, sofern diese Komponente nicht mit einer Fläche des Fixbüschels zusammenfällt. Hierfür wird ein Beispiel angegeben, bei dem die Fixpunktfläche eine Ebene ist und das Büschel aus Flächen 4. Ordnung besteht; ein anderes Beispiel betrachtet ein Büschel von Flächen 3. Grades, das nicht einem Ebenenbüschel birational äquivalent ist, wobei die Fixpunktfläche selber eine Fläche des Büschels ist. Nach weiteren Beispielen dieser Art handelt der letzte Paragraph dieses Kapitels von birationalen Transformationen mit einer Fläche oder Kurve von r-fach zählenden Fixpunkten. Auf einfache Weise ergibt sich mit Hilfe eines Büschels von Monoiden, dass es birationale Transformationen mit einer beliebig gegebenen Kurve von r-fach zählenden Fixpunkten gibt: für eine Fläche gilt dies nur, wenn diese zu einem Geradenbündel oder linearen Kongruenz von Kegelschnitten gehört. Die Dinge dieses letzten Kapitels lassen sich z. T. auf höhere Räume verallgemeinern. *W. Burau* (Hamburg).

Derwidué, L. Sur les transformations birationnelles cycliques. Bull. Soc. Roy. Sci. Liége 16, 31–37 (1947).

Diese Arbeit enthält im Anschluss an die vorhergehende desselben Verfassers [siehe vorstehendes Referat] noch einige Präzisierungen für den Fall der zyklischen ebenen Transformationen. Es wird zunächst ein Kurvensystem konstruiert, das bei einer zyklischen Transformation invariant bleibt, und das mit diesem System kovariant verbundene System der adjungierten, subadjungierten usw. Kurven betrachtet. Diese Adjunktionssysteme lassen sich auf gewisse Normalfälle birational transformieren und dadurch gelangt der Verfasser nach einigen Einzeldiskussionen und Fallunterscheidungen zu folgenden Sätzen: Die zyklischen Cremonatransformationen der Ebene lassen sich birational auf folgende 3 Typen zurückführen: (1) Transformationen, die ein Büschel von Geraden oder Kubiken in sich überführen, (2) Transformationen, die nur eine endliche Anzahl von Fixpunkten besitzen, (3) Transformationen, die die Geraden eines Büschels unter sich vertauschen und ausserdem eine oder zwei dieser Geraden als Fixpunktgebilde besitzen. *W. Burau* (Hamburg).

Differential Geometry

Arghiriade, E. Sur le contact d'une surface et d'une quadrique. Acad. Roum. Bull. Sect. Sci. **27**, 118–123 (1947).

Les quadriques surosculatrices en un point d'une surface, c'est-à-dire ayant un contact du troisième ordre avec la surface, sont, comme l'a montré Hermite, en général en nombre fini. En ce qui concerne les surfaces du troisième degré, Hermite [Cours d'Analyse de l'École Polytechnique, Gauthier-Villars, Paris, 1873, p. 146] affirme, sans démonstration, que les points en lesquels il existe des quadriques surosculatrices sont en nombre infini et distribués sur les 27 droites que contiennent ces surfaces. Dans cette note l'auteur montre que l'affirmation précédente est inexacte, et prouve que le nombre de points d'une surface du troisième degré en lesquels la surface admet une quadrique surosculatrice est en général fini, chaque droite de la surface contenant 12 points de surosculation (10 quadriques surosculatrices non dégénérées et deux décomposées en deux plans). Dans le cas particulier où la surface envisagée admet une droite de points paraboliques il existe, tout le long de la droite, un cône du deuxième degré surosculateur, et la surface n'admet pas d'autres cônes surosculateurs. D'une façon générale, si une surface, de degré quelconque, admet une ligne droite de points paraboliques, il existe en chaque point de cette droite un cône quadratique surosculateur.

P. Vincensini (Besançon).

Creangă, Ioan. Sur la relation entre les paramètres de distribution des surfaces réglées avec les génératrices respectivement parallèles. Acad. Roum. Bull. Sect. Sci. **25**, 56–60 (1943).

Cette note étudie les couples de surfaces réglées S_0 et S entre lesquelles existe une correspondance associant à une génératrice quelconque de S_0 une génératrice parallèle de S, le plan de deux génératrices homologues étant le plan osculateur d'une courbe gauche quelconque Γ_0 et la génératrice rectiligne de S_0 passant par le point d'osculation. L'auteur établit une formule exprimant le paramètre de distribution relatif à une génératrice quelconque de S au moyen de la courbure et de la torsion de Γ_0 au point correspondant et des éléments fixant S par rapport à Γ_0. La développable d'arête de rebroussement Γ_0 étant évidemment circonscrite à S, la formule indiquée exprime le paramètre de distribution d'une surface réglée quelconque S au moyen de la courbure et de la torsion des arêtes de rebroussement de ses développables circonscrites. Si l'on considère la développable circonscrite à S le long de sa ligne de striction, Γ_0 est ligne de striction pour S_0, la formule générale se simplifie et donne le paramètre de distribution d'une surface réglée quelconque S, en fonction, des rayons de courbure et de torsion de l'arête de rebroussement de la développable circonscrite le long de la ligne de striction, et de la distance des points correspondants sur l'arête de rebroussement et la ligne de striction.

P. Vincensini (Besançon).

Santaló, L. A. A characteristic property of the quadrics of revolution and of the cylinders whose cross section is a logarithmic spiral. Math. Notae **7**, 81–90 (1947). (Spanish)

Dans cette note il est question des courbes D d'une surface S telles qu'en chacun de leurs points la sphère osculatrice est tangente à S. Par chaque point M de S il passe ∞^1 courbes D (dépendant d'une équation différentielle du 2ème ordre), et le lieu géométrique des centres de courbure en M de ces ∞^1 courbes D est en général une courbe gauche. L'objet du présent travail est la recherche des surfaces S pour lesquelles, en chaque point régulier (courbures principales finies et dérivables) la courbe gauche précédente est plane. Le résultat obtenu se traduit par le théorème suivant: la sphère et le plan étant exclus, les seules surfaces jouissant de la propriété indiquée sont les quadriques de révolution et les cylindres dont la section droite est une spirale logarithmique.

Ne retenant que la propriété locale caractérisant les courbes D d'une surface, l'auteur caractérise les surfaces cherchées par le fait, que si l'on considère les ∞^1 sphères tangentes en l'un quelconque de leurs points, le lieu géométrique des centres de courbure en ce point des courbes d'intersection de la surface et des ∞^1 sphères indiquées est une courbe plane. La démonstration du théorème énoncé utilise l'équation d'une surface au voisinage de l'un de ses points rapportée au trièdre principal relatif à ce point, et les développements des coordonnées d'un point d'une courbe de la surface suivant les puissances de l'arc. La caractérisation des surfaces annoncée repose sur la considération des relations différentielles liant les courbures principales d'une quadrique et d'une surface de révolution quelconques.

Dans une note antérieure [Math. Gaz. **30**, 141–143 (1946)], E. D. Camier a étudié le problème des surfaces telles qu'en tout point le lieu géométrique des centres des sphères osculatrices des lignes géodésiques issues de ce point est une courbe plane. Santaló identifie ce problème avec celui qui fait l'objet de sa propre recherche, et cela lui permet de rectifier une conclusion de Camier. Ce dernier énonce ce résultat que tous les cylindres sont solutions du problème précédent; en réalité ne sont solutions que les cylindres de révolution et ceux ayant pour section droite une spirale logarithmique.

P. Vincensini (Besançon).

Backes, Fernand. Théorèmes sur les quadriques et les cyclides à lignes de courbure circulaires. C. R. Acad. Sci. Paris **226**, 1239–1240 (1948).

Let $u=$constant, $v=$constant be the generators of a quadric surface. Let these coordinates be transformed by $U=f(u)$, $V=g(v)$. It is proved that the lines joining the points with parameters (u, v) and (U, V) generate a W congruence, u and v being the asymptotic parameters on the focal surfaces.

V. G. Grove (East Lansing, Mich.).

Lalan, Victor. Sur l'équation de Gauss des surfaces d'Ossian Bonnet de troisième classe. C. R. Acad. Sci. Paris **226**, 777–779 (1948).

By a suitable change of variable the differential equations of a surface of O. Bonnet are transformed into the equations of Gauss for (1) a real or imaginary helicoid, or (2) a cylinder or a surface of revolution according to the type of the given surface. From a change of variable suggested by the above result first integrals are obtained for the differential equations of these surfaces.

C. B. Allendoerfer.

Lalan, Victor. Sur la détermination des surfaces à courbure moyenne isotherme. C. R. Acad. Sci. Paris **226**, 1339–1341 (1948).

In his theory of surfaces of isothermal mean curvature the author previously [Bull. Soc. Math. France **75**, 63–88 (1947); these Rev. **9**, 375] made use of the isothermal lines of the surface to introduce a function ψ. This approach required him to introduce complex quantities in his argument. In the present paper he uses purely real methods to

arrive at the same function ψ and the differential equation satisfied by it. *C. B. Allendoerfer* (Princeton, N. J.).

Bogdan, C. P. Sur les lignes asymptotiques de la surface de Steiner. Acad. Roum. Bull. Sect. Sci. **25**, 53–55 (1943).

Cette note donne une nouvelle démonstration du fait que les lignes asymptotiques d'une surface de Steiner sont des quartiques de deuxième espèce (quartiques rationnelles), et que l'une quelconque de ces courbes détermine une (et une seule) surface de Steiner. Dans sa démonstration l'auteur envisage une surface de Steiner comme la projection, sur un S_3, d'une surface de Veronese d'un S_5 dans lequel S_3 est supposée plongée. La projection est faite à partir d'une droite d de S_5, d ne coupant pas S_3 et n'appartenant pas à la variété des cordes de la surface de Veronese.

P. Vincensini (Besançon).

Rollero, A. L'intorno del 5° ordine di un punto di una superficie nello spazio proiettivo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 311–313 (1947).

The purpose of this note is to compute the coefficients of the expansion of a surface to terms of the fifth degree arising from the Wilczynski canonical form of the defining differential equations. Two of these six coefficients are invariant (outside of the projective transformations) under projective applicability. *V. G. Grove* (East Lansing, Mich.).

Maxia, Angelo. Studio proiettivo differenziale di un elemento cuspidale di specie superiore. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 253–345 (1947).

The purpose of this paper is the study of curves at a cuspidal point by means of so-called cuspidal elements at the point. If the origin is chosen at the point, the analytic representation of a cuspidal element is $x^m=\sum_{r,s}a_{rs}x^ry^s$, $rs>0$, $r+s>m\geqq 2$. The cuspidal element may be made to assume the parametric form $x=t^n(\alpha_0+\alpha_1 t+\cdots)$, $y=t^m$, $nr+ms\geqq mn$. The first part is devoted to general properties of such an element. Let S be the greatest common divisor of m, n. The integer m is said to be the order or multiplicity and $q=n-m$ the class of the element. Assigning values of $\alpha_0, \alpha_1, \cdots, \alpha_p$, $(p=(i-1)\delta+j+1)$, the element is said to have the symbol $(m, q, i\delta+j)$. Some differential invariants are found for two or more cuspidal elements. For example, the elements $x^m=ay^n$, $x^m=by^n$, have the projective invariant a/b. But $x^m=ay^n$, $x^p=by^q$, $m<n$, $p<q$, has no invariant unless $m/n=p/q$. However the set of three elements $x^m=ay^n$, $x^p=by^q$, $x^r=cy^s$, $m<n$, $p<q$, $r<s$ always possesses an invariant for all m, n, p, q, r, s. [Hsiung gave a more general derivation of such invariants; see Quart. J. Math., Oxford Ser. **18**, 129–132 (1947); these Rev. **9**, 157.] Especial attention is paid to the case in which m, n are relatively prime $(\delta=1)$, and to the element of symbol (m, q, i), $i<n+q$.

The second part is devoted to the study of cuspidal elements by means of their polar curves. Especial attention is paid to the symbol $(m, q, m+1)$, $m+1=n$. Projective invariants of the element are found and geometrical characterizations given in terms of cross ratios involving the cuspidal tangents, the sets of lines projecting the inflexions, the polars of the cuspidal tangent and those sets of lines.

Part three concerns itself with the study of cuspidal elements by means of approximating curves. Particular attention is paid to the symbol (m, q, i), $m>q\geqq 1$, projective invariants again being given and characterized in terms of cross ratios involving the cuspidal tangent, the inflexions of the approximating curves and the given element.

V. G. Grove (East Lansing, Mich.).

Gheorghiev, G. Sur les surfaces dont une famille de lignes de courbure sont des courbes planes semblables. Acad. Roum. Bull. Sect. Sci. **24**, 85–88 (1943).

Cette note a trait à la recherche des surfaces admettant une famille de lignes de courbure formée de courbes semblables. A. Myller [voir Ann. Sci. Univ. Jassy. Sect. I. **27**, 127–135 (1941); ces Rev. **8**, 345] a déterminé les surfaces possédant une famille de lignes de courbure homothétiques planes dont les plans forment un faisceau. L'auteur étudie le problème général où les lignes de courbure (semblables) de la famille considérée sont tracées dans les plans tangents d'une développable quelconque. Se donnant arbitrairement l'arête de rebroussement Γ de la développable, il ramène la détermination des courbes semblables à distribuer dans les différents plans osculateurs de Γ pour obtenir une surface du type envisagé, à l'intégration d'une équation différentielle du premier ordre qu'il interprète géométriquement, et dont les courbes intégrales admettent des points de rebroussement situés sur une certaine conique. Le résultat obtenu reste valable dans le cas particulier ou Γ est une droite, et redonne les courbes Smi de Myller [loc. cit.] lorsque la similitude des lignes de courbure se réduit à une homothétie.

Dans le cas général (Γ quelconque) les courbes intégrales de l'équation fournissant les lignes de courbure sont les trajectoires isogonales sous un certain angle (défini géométriquement) d'une certaine famille de courbes Smi. Les trajectoires orthogonales de cette famille de courbes Smi sont, comme l'auteur l'indique, des développantes aréolaires de la conique des rebroussements signalée plus haut par rapport à l'un de ses foyers. En ce sens, l'auteur appelle développoïdes aréolaires de la même conique les courbes planes semblables fournissant une famille de lignes de courbure distribuées dans les plans osculateurs de Γ.

P. Vincensini (Besançon).

Gheorghiu, Gh. Th. Une interprétation géométrique du faisceau canonique d'une surface. Acad. Roum. Bull. Sect. Sci. **28**, 88–90 (1945).

L'auteur associe d'abord à une quartique (C) à point triple à tangentes distinctes un faisceau linéaire de droites qu'il appelle droites principales de la courbe. Ces droites, parmi lesquelles se trouve la droite de Togliatti, sont les polaires du point triple par rapport aux coniques d'un faisceau linéaire défini projectivement à partir de la courbe C. Soit ensuite une surface (S) rapportée à un tétraèdre dont deux arêtes soient les tangents asymptotiques et soient P_1, P_2 les plans des faces autres que le plan tangent, qui contiennent les tangents asymptotiques; dans le système linéaire défini par le cône de Segre et les deux cônes décomposés formés par les plans P_1, P_2 comptés chacun six fois, existe un seul cône qui se décompose en un cône de quatrième classe et les plans P_1, P_2 [G. Fubini et E. Čech, Geometria Proiettiva Differenziale, Zanichelli, Bologna, 1926, § 24]. La figure corrélative de ce cône est une quartique C à point triple dont le faisceau des droites principales est le faisceau canonique de la surface contenu dans le plan tangent; les diverses droites canoniques remarquables sont caractérisées par des propriétés géométriques des coniques auxquelles elles sont associées. Ces résultats sont énoncés, mais non démontrés dans la note présente. *M. Decuyper* (Lille).

Grove, V. G. Generalized canonical lines. Duke Math. J. 15, 95–103 (1948).

En géométrie projective différentielle, on associe à tout point d'une surface certaines droites remarquables comme les directrices de Wilczynski, les arêtes de Green, les axes de Čech, la normale projective; elles appartiennent aux faisceaux canoniques. L'auteur donne une méthode permettant de trouver d'autres droites canoniques qu'il appelle "droites canoniques généralisées." Pour cette construction, il utilise des courbes covariantes et des points covariants: "points générateurs intégraux." Les droites canoniques classiques peuvent être retrouvées par ce procédé, la courbe covariante étant alors une quelconque des courbes de Segre. Enfin, en utilisant une méthode voisine de celle de Wilkins [même J. **10**, 173–178 (1943); ces Rev. **5**, 14] l'auteur montre comment une droite canonique généralisée peut être définie par un couple d'invariants intégraux. *M. Decuyper.*

Čahtauri, A. I. On canonical bundles of lines. Doklady Akad. Nauk SSSR (N.S.) **59**, 1257–1259 (1948). (Russian)

L'auteur donne, du faisceau canonique d'une surface ou d'un réseau plan, une expression analytique, possédant, sur celle de Fubini et Čech, l'avantage d'être indépendante, s'il s'agit d'une surface, du choix du tétraèdre mobile et de sa normalisation; s'il s'agit d'un réseau plan, du choix des coordonnées curvilignes et de la normalisation du tenseur de ce réseau. Pour les notations, se reporter aux traités de Fubini et Čech, aux notes de Norden [C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 625–628 (1945); **53**, 495–498 (1946); ces Rev. **8**, 93, 346], à un article de Dubnov sur l'analyse des vecteurs et tenseurs [Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] **4**, 197–202 (1937)]. Pour une surface de l'espace projectif à 3 dimensions, normalisée au sens de Norden, soient x^α les coordonnées du point courant, ξ^α celles du plan tangent en ce point: les points $y_i^\alpha=\partial_i x^\alpha-l_i x^\alpha$ définissent la normale de seconde espèce, les plans $\eta_{\alpha i}=\partial_i\xi_\alpha-l_i\xi_\alpha$ la normale de première espèce; G_{ij}^m, Γ_{ij}^m sont les coefficients de la connexion intrinsèque de première et seconde espèce; R_{ij} tenseur de Ricci de la connexion de première espèce, b_{ij} tenseur du réseau asymptotique, t_i tenseur de Tchebitcheff du réseau asymptotique de la connexion G_{ij}^m; pour une surface normalisée, on définit les tenseurs τ_i, v_i indépendants de la normalisation des b_{ij}:

$$\tau_i=-\frac{q^{ks}(\nabla_i b_{ks}-t_k b_{is})}{2J},\qquad v_i=\tfrac{1}{3}\tilde{b}^{ks}(\nabla_i b_{ks}-2t_k b_{is})+\partial_i\log J,$$

avec

$$J=\tilde{b}^{ij}\tilde{b}^{ks}\tilde{b}^{mn}(\nabla_i b_{km}-t_k b_{im})(\nabla_j b_{sn}-t_s b_{jn}),$$

$$q^{ij}=\tilde{b}^{ik}\tilde{b}^{js}(\tilde{R}_{(ks)}-\tfrac{1}{3}\tilde{b}^{mn}\tilde{R}_{(mn)}b_{ks}),\qquad \tilde{R}_{(ks)}=R_{ks}-\tfrac{1}{2}\nabla_{k+s}+\tfrac{1}{4}t_k t_s.$$

Une droite arbitraire du faisceau canonique est l'intersection des plans $\eta_{\alpha i}=\eta_{\alpha i}+q_i\xi_\alpha$, avec $q_i=-\lambda\tau_i+(\lambda+\frac{1}{3})v_i$, λ étant le paramètre du faisceau; $\lambda=-\frac{1}{3}$ donne la directrice de Wilczynski; $\lambda=0$, la normale de Fubini. La droite réciproque, par rapport à la quadrique de Lie, d'une droite du faisceau canonique sera appelée droite canonique de seconde espèce et est définie par les points $z_i^\alpha=y_i^\alpha+px^\alpha$ où $p_i=\frac{1}{2}t_i+q_i$. Cette définition se conserve pour un réseau plan quelconque, à condition de remplacer b_{ij} par le tenseur d'un réseau plan arbitraire.

Partant de la conception de la droite de Laplace relative à un réseau plan, nous arrivons par dualité à la conception du point de Laplace pour une congruence plane de droites. La droite réunissant les points de Laplace de la congruence des tangentes aux courbes du réseau s'appelle seconde droite de Laplace: la représentation de cette dernière se fait par les points $z_i^\alpha=y_i^\alpha+\tau x^\alpha$, tandis que la première droite de Laplace est définie par les points y_i^α. Les deux droites de Laplace, la droite joignant leur point d'intersection au point x^α et la directrice de Wilczynski forment un faisceau harmonique. Cette méthode permet de retrouver tous les résultats de Fubini et Čech pour le faisceau canonique d'un réseau plan. *B. Gambier* (Paris).

Decuyper, Marcel. Sur quelques congruences attachées à une surface. J. Math. Pures Appl. (9) **26**, 15–98 (1947).

This memoir is composed of three parts. The first part is devoted to the study of the first and second axis of a conjugate net N (that is, to the axis and ray of the net). Among the ideas discussed is the condition that the congruences of axes and rays be stratifiable in both senses, that is form a "stratifiable pair," or in the language of Cook [Trans. Amer. Math. Soc. **32**, 31–46 (1930)] each congruence has the intersector property with respect to the other. These conditions are applied to the sequence of Laplace generated by N, and as a typical theorem may be quoted the following. If, in a sequence of Laplace, the congruences of axes and rays of one net of the sequence form a stratifiable pair, then the same is true of all nets of the sequence. The conditions that the axes of consecutive conjugate nets of a Laplace sequence be a stratifiable pair are found; the results are embodied in the following theorem. Let consecutive conjugate nets of a Laplace sequence be generated by points M, N, and suppose that the axes of these nets form a stratifiable pair; then the congruence of lines MN belong to a particular class of congruences which are W-congruences and also congruences of Guichard. Finally a study is made of pairs of conjugate nets on surfaces having the same axes. It is found that such pairs of surfaces depend on six arbitrary functions of one variable.

The second part concerns itself with two problems. (1) Given a congruence C and a surface S in one-to-one correspondence the lines of C passing through the corresponding points of S but not lying in the tangent planes at those points, does there exist on S a conjugate net having C as its axis? (2) Given a congruence C, can a surface S be found having one (or more) conjugate nets having C as its (or their) axis congruence? In the first problem it is found that for a given surface S and congruence C then S may sustain zero, one, two or an infinity of solutions. However, on every surface S one may associate an infinity of congruences C each of which may serve as axis congruence of two distinct conjugate nets on S; and on each surface S there is an infinity of pairs of conjugate nets, depending on one arbitrary function of one variable, such that the two nets have the same axis congruence. With respect to the second question the following theorems may be quoted. Every congruence C may serve as the axis of a conjugate net N whose sustaining surface S depends on four arbitrary functions of one variable. In particular, if C is linear, the net N is formed by curves belonging to linear complexes with respect to which the directrices of C are conjugate. In this case some solutions are exhibited depending only on three arbitrary functions of one variable on each surface of which there is a family of conjugate nets formed by curves belonging to the same linear complex, and also some solutions depending on two functions of one variable, on each

surface of which there are two families of conjugate nets formed by curves belonging to the same complex.

The third part of the memoir is devoted to a study of the directrices of Wilczynski. A typical theorem may be stated as follows. A focal point of either directrix is the pole with respect to the quadric of Lie of one of the focal planes of the other directrix. This result is used to study pairs of surfaces having the same first and second directrices, and pairs for which the first and second directrices of one surface serve as the second and first directrices of the other. Finally, the directrices of Wilczynski are found for a surface whose equation is written in the Monge form $z=f(x, y)$.

The method used in the first and third parts is that of defining linear differential equations of the first order, that is, the so-called method of moving frames of reference of Cartan, the local tetrahedron of reference being chosen in a manner convenient to the problem. In particular, in the third part the so-called normal tetrahedron of Cartan is used. In the second part the geometrical entities are given in terms of a fixed coordinate system, that is, the method may be considered as the classical method. Rectangular coordinates are assumed and, although the problems of this part are projective in nature, some metrical (or affine) examples are given. *V. G. Grove* (East Lansing, Mich.).

Roşca, Radu. Sur les courbes et variétés linéaires dans un espace de courbure constante. Acad. Roum. Bull. Sect. Sci. **24**, 625–633 (1943).

The author considers systems of spheres and straight lines in non-Euclidean spaces I_n. A circle in an elliptic space I_2 with center (a^1, a^2, a^3) and radius r may be represented by a point in an I_3 with coordinates $(a^1, a^2, a^3, i\cos r)$. Two points conjugate with respect to the absolute quadric in I_3 correspond to two orthogonal circles. From this it follows that a pencil of circles is represented by a straight line in I_3. A straight line in I_3 is represented by a point on a quadric in an I_5 (coordinates of Klein). By means of this correspondence the author studies the complexes, the congruences (particularly the congruences of Clifford) and the one-parameter systems of linear complexes (a curve in I_5). *J. Haantjes* (Amsterdam).

Popa, Ilie I. Sur les réseaux plans. Acad. Roum. Bull. Sect. Sci. **23**, 31–33 (1942).

Let there be given a plane net $dudv=0$. Let $x(u, v)$ be a point in the plane and $y=x(u+\Delta u, v+\Delta v)$ be a second point. The parametric tangents to $u=$constant and $v=$constant at x and y intersect in points whose limits as y approaches x determine a line h. The correspondence between h and the line $l(x, y)$ is such that to l corresponds one line h, and to a line h there corresponds one line l. The envelope of h as l varies in the pencil on x is a conic. Using this correspondence a characterization is given of the Darboux-Segre pencil in the plane. *V. G. Grove* (East Lansing, Mich.).

***Vranceanu, G. Leçons de Géométrie Différentielle. Vol. 1. Congruences. Formes de Pfaff. Groupes continus. Invariants et équivalence. Espaces à connexion affine. Espaces de Riemann. Espaces à connexion projective.** Bucarest, 1947. 422 pp.

This is a thoroughly modern and scholarly treatise on differential geometry presented with considerable originality and seasoned with the author's nonholonomic predilections. It is meant primarily for the specialist. The style is rather informal and discursive. Whenever possible, a naive attitude is assumed toward the subject matter; in other words, simplicity is preferred to elegance, hammer and tongs methods to clever devices. The reader benefits from this approach. The methods of both Ricci and Cartan (tensors and Pfaffians) are used, with ennuples the connecting link. Chapter I. Congruences, Pfaffians. This includes the simpler properties of tensors; Pfaffians, bilinear covariants, and their canonical forms; systems of congruences (ennuples) and their calculus. Chapter II. Finite continuous groups. This is quite a complete and unhackneyed presentation from the Lie-Cartan viewpoint including even such topics as the canonical forms of the structure tensor of any G_3 and the derivation of all primitive groups in two variables. Chapter III. Invariants and equivalence. The Cartan approach through Pfaffian systems is developed and applied to several geometric and analytic problems such as invariants and equivalence of webs (textile geometry) and of ordinary differential equations of first and second orders. It concerns an aspect of Pfaffian theory which is not widely known and even less widely understood. Fortunately the author spares no details. Chapter IV. Affinely connected spaces. The approach is through the notion of affine displacement of the spaces of differentials. Besides the standard topics there is an unusually thorough treatment of torsion and curvature tensors and their interrelations. Also notable is the introduction and application of the concept of group associated with an affinely connected space. The methods of chapter III are applied to the problem of equivalence and automorphism. There is a detailed discussion of the homogeneous spaces of Cartan. Then for good measure comes a long coda on Cartan's structure equations. Chapter V. Riemann spaces. In the short space of one chapter the author hits the high spots of this familiar discipline with a stimulating combination of eclecticism and originality. A short, novel treatment of subspaces of Euclidean space is included. Chapter VI. Projectively connected spaces. In this more controversial topic the author takes the Cartan-Veblen standpoint using for the most part Veblen's formalism. After introduction of projective ennuples a transition to the Cartan formalism is exhibited. The more special projective geometries of Weyl and of T. Y. Thomas are oriented within the general theory. Some of the developments of chapter IV are extended to the present case. An unusually comprehensive presentation results. The proposed second volume is to include conformally connected spaces, holonomic and nonholonomic subspaces and invariants of second order partial differential equations.

J. L. Vanderslice (College Park, Md.).

Blum, Richard. Anzahl der Identitäten von Bianchi in einer V_n. Acad. Roum. Bull. Sect. Sci. **28**, 351–353 (1946).

It is shown that the number of independent Bianchi identities for an n-dimensional Riemann curvature tensor is $n^2(n^2-1)(n-2)/24$. *A. Fialkow* (Brooklyn, N. Y.).

Théodoresco, N. Géodésiques de longueur nulle et propagation des ondes. Acad. Roum. Bull. Sect. Sci. **23**, 132–137 (1942).

Rappel de la théorie classique des géodésiques de longueur nulle d'un espace de Riemann à métrique indéfinie. Ce papier est destiné à préparer le suivant.

A. Lichnerowicz (Strasbourg).

Théodoresco, N. Géométrie finslérienne et propagation des ondes. Acad. Roum. Bull. Sect. Sci. 23, 138–144 (1942).

Extension des résultats rappelés ci-dessus aux géodésiques de longueur nulle d'un espace de Finsler à cône fondamental réel. *A. Lichnerowicz* (Strasbourg).

Castoldi, Luigi. Sopra gli spazi a connessione semimetrica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 6, 395–409 (1947).

It is proved that the third identity of the curvature affinor in an L_n can be written in the form

$$R_{\nu\mu}(\lambda\rho)=2\overset{R}{\nabla}_{[\nu}Q_{\mu]\lambda\rho},$$

where $\overset{R}{\nabla}$ denotes the covariant differentiation with respect to $a_{\lambda\varepsilon}$ and $Q_{\mu\lambda\varepsilon}=-\nabla_\mu a_{\lambda\varepsilon}$. If $N=v_\lambda v^\lambda$ and $\omega=v_\lambda w^\lambda$ it is proved that the differences ΔN, $\Delta\omega$ between initial value and final value after pseudoparallel transplantation along $2d\xi^{[\mu}_1 d\xi^{\lambda]}_2$ can be written

$$\Delta N=2(2\overset{R}{\nabla}_{[\mu}Q_{\rho]\chi\lambda}-S^{\cdot\cdot\tau}_{\mu\rho}\,Q_{\tau\chi\lambda})v^\chi v^\lambda d\xi^\mu_1 d\xi^\rho_2,$$

$$\Delta\omega=2(2\overset{R}{\nabla}_{[\mu}Q_{\rho]\chi\lambda}-S^{\cdot\cdot\tau}_{\mu\rho}\,Q_{\tau\chi\lambda})v^\chi w^\lambda d\xi^\mu_1 d\xi^\rho_2.$$

From this follows the (well-known) result that the transformation after the transplantation is a rotation with a similarity if and only if the connection in L_n is semimetric. In L_n there exists a system of ∞^p X_{n-p}'s such that pseudoparallel transplantation along every curve in an X_{n-p} conserves length if and only if the class of Q_λ is $2p$ or $2p-1$.

With respect to the question whether X_m's in L_n exist such that pseudoparallel transplantation along every closed curve in an X_m conserves (at the end) length, some partial results are given. A necessary and sufficient condition is that for every 2-dimensional facet $f^{\mu\nu}d\sigma$ in X_m the equation $f^{\mu\lambda}\nabla_{[\mu}Q_{\lambda]}$ holds. [This is a Goursat problem that can be solved by the methods of Cartan and Kähler; cf. chap. 7 of Schouten and van der Kulk's forthcoming book on Pfaff's problem.] The kernel-index method is used.

J. A. Schouten (Epe).

Morinaga, Kakutaro. Mathematical foundation of wave geometry. II. A generalization of Clifford number. J. Sci. Hirosima Univ. Ser. A. 11, 137–170 (1942).

This paper is a continuation of the discussion by the same author [same J. Ser. A. 10, 215–246 (1940); these Rev. 2, 166] of a proposed generalisation of wave geometry, called an iteration system. In this paper matrix representations of iteration systems are found. In addition automorphisms are studied. *A. H. Taub* (Princeton, N. J.).

NUMERICAL AND GRAPHICAL METHODS

***Tables of Bessel Functions of Fractional Order. Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. Volume I.** Columbia University Press, New York, 1948. xlii+413 pp. $7.50.

Bessel functions of fractional order occur in many problems of elasticity, heat conduction, wave propagation in stratified media, hydrodynamics, and other branches of applied mathematics. They are also important for the approximate solution of differential equations of the second order of the form $y''(x)+p(x)y(x)=0$ in the neighbourhood of a zero of $p(x)$. Functions whose order is the half of an odd integer have already been tabulated [Tables of Spherical Bessel Functions; cf. these Rev. 8, 406; 9, 103]; the present work is concerned with Bessel functions of orders $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, $\pm\frac{3}{4}$, $\pm\frac{3}{4}$.

The functional values are given to 10 decimal places or to 10 significant figures. The interval of x has been so chosen that interpolation with the aid of the tabulated central differences (modified where necessary) may yield the maximum attainable accuracy. For very small x, interpolation of the Bessel functions themselves is not feasible, and here auxiliary functions have been tabulated. Again, for large values of x, the functions $A_\nu(x)$ and $B_\nu(x)$ of the asymptotic expansion

$$J_\nu(x)\sim A_\nu(x)\cos(x-\tfrac{1}{4}(2\nu+1)\pi)-B_\nu(x)\sin(x-\tfrac{1}{4}(2\nu+1)\pi)$$

have been tabulated. Interpolation in the ν direction, for fixed x, is based on Lagrange's formula $J_\mu(x)=\sum_\nu L_\nu(\mu)J_\nu(x)+R$, where ν ranges over the eight tabulated values. For $-\frac{3}{4}<\mu<\frac{3}{4}$ this gives at least 3 to 4 place accuracy for $0.1\leqq x\leqq 0.5$ and at least 5 to 6 place accuracy for larger values of x. With a slight loss of accuracy, the formula may be used for $-1<\mu<1$.

The introduction by M. Abramovitz gives the definition of and many formulae regarding $J_\nu(x)$ and $I_\nu(x)$ as well as other Bessel functions, connection of the functions of order $\pm\frac{1}{3}$ with Airy integrals, differential equations that can be solved in terms of Bessel functions, notes on the zeros of these functions, interpolation in the x direction for fixed ν by Everett's formula, interpolation in the ν direction for fixed x by Lagrange's formula [the developments in this section are due to H. E. Salzer], and a description of the method of computation. There is an additional note on modified second differences for use with Everett's interpolation formula, and a bibliography of 30 items.

Tables: $J_\nu(x)$ for $x=0(.001)X(.01)25$ to 10 D, where $X=0.9$ for $\nu=-\frac{3}{4}$, $-\frac{3}{5}$; $X=0.8$ for $\nu=-\frac{1}{3}$, $-\frac{1}{4}$; $X=0.6$ for $\nu=\frac{1}{4}$, $\frac{1}{3}$; and $X=0.5$ for $\nu=\frac{2}{3}$, $\frac{3}{4}$. $A_\nu(x)$ and $B_\nu(x)$ for $x=25(.1)50(1)500(10)5000(100)10000(200)30000$ to 10 D; ν has the same values as above. The first 30 zeros, to 10 D, of $J_\nu(x)$ for the above values of ν with approximate formulae for the computation to at least 10 decimal accuracy of the later zeros. Everett interpolation coefficients for second and fourth order central differences. Lagrangian interpolation coefficients $L_\nu(\mu)$ for the eight base points $\pm\nu=\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$ and for $\mu=0(.001)1$. Multiples of $\pi/2$ and various constants.

The second volume will contain similar tables for modified Bessel functions. *A. Erdélyi* (Edinburgh).

Thomas, D. E. Tables of phase associated with a semi-infinite unit slope of attenuation. Bell System Tech. J. 26, 870–899 (1947).

This paper gives tables of the function

$$B(x)=\pi^{-1}\int_0^x t^{-1}\log_e|(1+t)/(1-t)|\,dt$$

which is related to Spence's integral,

$$L(u)=\int_1^u (u-1)^{-1}\log_e u\,du.$$

[See a review of an earlier less-extended table by M. S. Corrington [RCA Rev. 7, 432–437 (1946); these Rev. 8, 534].] Values of $B(x)$ and of $(180/\pi)B(x)$ are given to 5

figures, for values 0(.001)0.996(.0005)0.998(.0001)1 of x and of $1/x$. An appendix illustrates the use of the tables in determining phase from attenuation or reactance from resistance. *J. C. P. Miller* (London).

LeCaine, J. A table of integrals involving the functions $E_n(x)=\int_1^\infty e^{-xu}u^{-n}du$. National Research Council of Canada. Division of Atomic Energy. Document no. **MT-131** (NRC 1553), i+45 pp. (1948).

The functions $E_n(x)$ which are defined in the title and are closely connected with the incomplete gamma function have been investigated, and to some extent tabulated, by G. Placzek [same series, no. **MT-1** (1946); these Rev. **9**, 159]. The present publication is a collection of formulae for, or useful in connection with, these functions. [Cf. also V. Kourganoff, C. R. Acad. Sci. Paris **225**, 430–431, 451–453 (1947); Ann. Astrophysique **10**, 282–299, 329–340 (1947); these Rev. **9**, 91, 92, 349, 432, for some of the formulae.] There are integrals involving products of two or more factors e^{kx}, $e^{-k|x-y|}$, x^m, $E_n(kx)$, $E_n(k|x-y|)$, $E_n(ax+b)$; also some double integrals. The inclusion of both definite and indefinite integrals in the table is a very useful feature. Supplementary matter includes some useful algebraic formulae and expansions, definite integrals involving $\log(x+k)$, and the Laplace transform of $\log^n x$. In all these formulae x is the variable of integration (x and y in double integrals).

A. Erdélyi (Edinburgh).

Vilner, I. A. Diagrams for calculating the hyperbolic and circular tangents and cotangents of complex argument. Akad. Nauk SSSR. J. Appl. Math. Mech. N.S. **4**, no. 1, 145–152 (1940). (Russian. English summary)

An alignment diagram is presented which permits determination of any two of the four quantities in the relation $\tanh(a+bi)=re^{i\theta}$ when the other two are given. The four scales fall on four straight lines. Their disposition very closely resembles the nomogram for $\tan z$ given by H. Schwerdt [Die Anwendung der Nomographie in der Mathematik, Springer, Berlin, 1931, figure 235]. The equations connecting the variables are those of J. Rybner [General Electric Rev. **33**, 164–179 (1930)]. Although the diagram is a projective transformation of Rybner's diagram, it is shown to be a real improvement over his since the ranges of the variables that are included permit solution for any values. The scales are also calibrated so they can be used for $\tan(\alpha+i\beta)=re^{i\varphi}$. *R. Church* (Annapolis, Md.).

Vilner, I. A. Sur les nomogrammes des systèmes d'équations et des fonctions analytiques. Akad. Nauk SSSR. J. Appl. Math. Mech. N.S. **4**, no. 2, 105–116 (1940). (Russian. French summary)

Necessary and sufficient conditions are given for the possibility of constructing an alignment chart with two straight scales (for x and y) for the system of equations $f_1(x, y, z_1)=0$, $f_2(x, y, z_2)=0$, so that any two variables can be determined when the other two are given. The conditions are $\Delta_{1y}=\Delta_{2x}=0$, where

$$\Delta_1=\frac{\tau_1\tau_2[\ln(\tau_1/\tau_2)]_y+\tau_1[\ln\tau_2]_x-\tau_2[\ln\tau_1]_x}{\tau_2-\tau_1},$$

$$\Delta_2=\frac{[\ln(\tau_2/\tau_1)]_x+\tau_1[\ln\tau_1]_y-\tau_2[\ln\tau_2]_y}{\tau_2-\tau_1}$$

and $\tau_1=f_{1x}/f_{1y}$, $\tau_2=f_{2x}/f_{2y}$, $f_{1x}f_{2y}-f_{2x}f_{1y}\neq 0$. The proof as outlined shows how to reduce to canonical form if the conditions are satisfied.

The method is applied to analytic functions of a complex variable, $w=f(z)$, where $z=a+ib$ and $w=p+iq$. It is shown that if both of the following conditions of the type of P. de Saint-Robert [Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (2) **25**, 53–72 (1871)] are satisfied,

$$\frac{\partial^2\ln\tau}{\partial p\partial q}=0,\quad \tau=\frac{a_p}{a_q};\qquad \frac{\partial^2\ln\tau}{\partial a\partial b}=0,\quad \tau=\frac{p_a}{p_b},$$

the nomogram is possible with straight scales for a, b, p, q; while if only the first (or second) is satisfied, the chart has a and b (or p and q) scales straight and the other two on a conic. An odd number of curved scales is impossible for an analytic function. Corresponding results are given for w in the form $re^{i\theta}$. The tests are applied briefly to eight functions as illustrations; one is the subject of the paper reviewed above. *R. Church* (Annapolis, Md.).

Vil'ner, I. A. Nomograms for systems of equations and analytic functions. Doklady Akad. Nauk SSSR (N.S.) **58**, 729–732 (1947). (Russian)

Using the method of binary fields the author outlines a general approach to the problem of nomographing a system of two equations. It leads to the following necessary and sufficient conditions [using the notation of the preceding review]:

$$(1)\qquad -\Delta_1=[\ln(\Delta_{2x}-2\Delta_{1y})]_x,\quad \Delta_2=[\ln(2\Delta_{2x}-\Delta_{1y})]_y.$$

If either (or both) of the round bracket expressions in (1) vanishes identically (the corresponding equation is then considered satisfied) the corresponding x or y scale (or both) is straight and conversely. If one (or both) of the expressions

$$(2)\qquad \Delta_{1y}+\Delta_{2x}-3(\ln\tau_1)_{xy},\quad \Delta_{1y}+\Delta_{2x}-3(\ln\tau_2)_{xy}$$

vanishes, the z_1 or z_2 scale (or both) is straight. The reduction to appropriate canonical form is possible but not given. A new proof of the conditions $\Delta_{1y}=0$ and $\Delta_{2x}=0$ for a system to be of genus 0, 1 or 2 (the genus is the number of curved scales) is outlined; it leads to a means of reducing the system to canonical form, when the conditions are satisfied, using only two quadratures. The author refers to two other ways of performing this in addition to that given in the first paragraph of the paper reviewed above.

The paper closes with a summary of the results of applying the criteria (1) and (2) to analytic functions of a complex variable. In general an analytic function which can be represented as a nomogram will have scales for the real and imaginary parts of w on one conic and those for z on another. Just as the functions given previously [C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 187–190 (1946); **55**, 783–786 (1947); these Rev. **8**, 494; **9**, 106] exhaust the analytic functions having nomograms of genus 2 and 0 (one or both conics degenerate), the analytic functions having nomograms of genus 4 are obtained by a process of elimination of z between two functions of second genus [to be published later]. *R. Church* (Annapolis, Md.).

Birge, Raymond T. Least-squares' fitting of data by means of polynomials. Mathematical appendix by J. W. Weinberg. Rev. Modern Physics **19**, 298–360 (1947).

This is a largely expository article explaining in great detail a numerical method of fitting least-squares polynomials to experimental data. A general method of obtaining the probable errors of the calculated values and of the coefficients in the polynomial is given for the first time.

T. N. E. Greville (Washington, D. C.).

Langebartel, Ray G. A note on Kepler's equation. Astr. J. 53, 101–104 (1948).

E. T. Whittaker obtained a development in the form of an infinite series of the root of a polynomial which is smallest in absolute value [Proc. Edinburgh Math. Soc. 36, 103–106 (1918)]. This result was extended to infinite series by Agnew, Rosser and Higgins [Higgins, Amer. Math. Monthly 49, 462–465 (1942); these Rev. 4, 90]. Their result, which is much more convenient for computational purposes, is: the root u_0 of $-1+a_1x+a_2x^2+a_3x^3+\cdots=0$, smallest in absolute value, is given by $u_0=\lim_{n\to\infty}(A_n/A_{n+1})$; $A_0=0$, $A_n=a_1A_{n-1}+\cdots+a_nA_0$ $(n>0)$. This method is applied to Kepler's equation: $u-e\sin u-M=0$, $0<e<1$, or:

$$-1+\frac{1-e}{M}u+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e}{M\cdot(2k+1)!}u^{2k+1}=0.$$

This application is illustrated by many examples. The convergence is most rapid when $N=M/(1-e)$ is small, particularly when $N<1$. If e is small but M is large, the convergence is improved by reducing M by some multiple of $\pi/2$. The critical case when e is close to 1 is surmounted by the transformation $u=u'+a$, $M=M'+a$, where a is an approximate root of Kepler's equation. *S. C. van Veen.*

Hanov, B. A kinematic solution of a trinomial equation. Trav. Inst. Math. Stekloff 20, 131–133 (1947). (Russian)

A plane in which the curve $f_1=r_1^n$ is plotted can be revolved through an angle θ about an axis, along which the r_1-axis lies, perpendicular to the (X, Y)-plane at the origin. A second plane containing the line $f_2=pr_2$ (actually a narrow beam of light) can be revolved through an angle ψ about an axis perpendicular to the (X, Y)-plane at the point $(q\cos\beta, q\sin\beta)$. The r_2-axis lies along this axis of revolution. By means of pulleys the angles are related by $\theta=n\varphi$ and $\psi=\varphi+\alpha$ (α constant). When the curves in the movable planes intersect, $r_1=r_2$ and the vector triangle for $z^n+Pz+Q=0$ is realized in the (X, Y)-plane, where $z=re^{i\varphi}$, $P=pe^{i\alpha}$ and $Q=qe^{i\beta}$. Details of construction are not given for the device built by the author and considered by him to be suitable for first approximations of the roots of the trinomial equation. *R. Church* (Annapolis, Md.).

Hardy, Arthur C., and Dench, Edward C. An electronic method for solving simultaneous equations. J. Opt. Soc. Amer. 38, 308–312 (1948).

The equations solved are the equations for three and four color printing processes. In the three color process the equations are $R(c, m, y)=R'$, $G(c, m, y)=G'$, $B(c, m, y)=B'$, where the left hand sides are cubics of a fixed form, and R', G', B' are functions of time generated by three photo tubes during a scanning operation. The problem is to find $c(t)$, $m(t)$ and $y(t)$. In the four color process the functions are quartics, and a fourth equation is adjoined to the system. It is said that the method of feedback using $R-R'$ to control c, $G-G'$ to control m, and $B-B'$ to control y ensures the finding of the desired solutions of the equations.

Multiplication is accomplished by using square pulse generators with irrationally related frequencies in the neighborhood of 18,000, 21,000, and 24,000 cycles per second, with the length of the positive pulse being proportional to the value of the variable. Regarding these as independent probabilities it follows that the common occurrence of two or three of them is proportional to their product (if one waits long enough). These combined pulses are fed into suitably biased tubes whose outputs are then the desired products. The authors say they "were pleased to discover that the errors in computation were significantly less than other errors that occur in color printing, even when dots were recorded at a rate of more than one thousand per second." *R. W. Hamming* (Murray Hill, N. J.).

Goldberg, Edwin A., and Brown, George W. An electronic simultaneous equation solver. J. Appl. Phys. 19, 339–345 (1948).

"An electronic device which will solve systems of equations up to ten in number has been constructed at RCA Laboratories. . . . The principles employed can be used to construct a device for solving systems of any number of simultaneous linear equations, and the same basic circuit design as employed in the ten equation solver can be used practically in devices for solving up to about twenty-five simultaneous equations." The basic principles used are simple; the coefficients of the system are set by hand on potentiometers, high gain summing amplifiers are used to calculate the errors in the equations, and the ith error is used to correct the ith variable. The values of the variables are measured one at a time by means of a balancing bridge. A cathode ray tube is included which detects both instability and overloading of the amplifiers. Special provisions are made for handling negative numbers. The questions connected with stability are carefully examined.

R. W. Hamming (Murray Hill, N. J.).

Tolstov, Yu. G. The conformal representation of doubly connected regions with the aid of an electric integrator. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1944, 447–461 (1944). (Russian)

The electrical integrator mentioned in the title is due to Gutenmaher and was more fully described in another paper unavailable to the reviewer. It evidently consists of a rectangular grid (16×13 in the present case) of resistances. Such a resistance grid represents an approximation to an electrolytic tank, and therefore permits the experimental solution of potential problems involving Laplace's equations, as in problems of conformal mapping. In the Gutenmaher integrator each of the component resistances is variable; this permits the solution of more difficult problems than could be carried out with an electrolytic tank. It is stated that comparisons of theoretical results and experimental results employing the present method show discrepancies of only 2 to 3 per cent. *H. Wallman.*

Tolstov, Yu. G. The use of an electric integrator for the conformal mapping of simply connected regions. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1947, 159–164 (1947). (Russian)

In the paper reviewed above the author discussed certain aspects of the problem of computing approximately the function mapping a given doubly-connected domain onto a ring. In this note he makes some remarks concerning the computation of the mapping of a given simply-connected domain onto the interior of a circle, the problem in this case being equivalent to computing the Green's function. If a small circle of radius r around the pole is excluded from the domain, this function has values $\log 1/r+O(1)$ on the circumference and the author gives an upper bound for r in order that the error introduced by neglecting the term $O(1)$ in the boundary values of the Green's function will not exceed an assumed error at the boundary of the domain.

D. C. Spencer (Stanford University, Calif.).

Kovner, S. S., and Žak, D. K. Calculation of the degrees of the operators of Liebmann and Geršgorin and their application to the mechanical integration of equations. Doklady Akad. Nauk SSSR (N.S.) **58**, 5–8 (1947). (Russian)

An improvement in the speed of convergence of methods for determining the degrees of the operators of Liebmann and Geršgorin is obtained by making use of derivative approximations based on differences of higher order. A systematic procedure, with tables, is proposed.

H. Wallman (Göteborg).

Byhovskiĭ, M. L. The accuracy of mechanisms controlled by differential equations. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 1455–1512 (1947). (Russian)

For the disc-and-wheel integrator 31 sources of error (including, for instance, lubrication film thickness) exist; for the associated screw-and-block 47 of them are listed. The first variation (called "the error" by the author) of the output integral is laboriously expressed in terms of the variations of the listed design parameters and the input variables. Of the original 78 variations only twelve turn out to be relevant, and enter into nine integrals of the form $I=\int_{v_0}^{v}\eta u dv$, where v is the variable of integration, and η a random function whose mean value $\bar{\eta}=\eta_i$ and dispersion $D\eta_i$ are constant within each of a set of intervals L_i of the same length Δx.

If the correlation coefficient between the random variables $\eta(x_1)$ and $\eta(x_2)$ is $r(x_1, x_2)$, then the author claims that

$$\bar{I}=\int_{v_0}^{v}\bar{\eta}u dv, \quad D_I^2=\int_{v_0}^{v}\int_{v_0}^{v} r(x, y)u(x)u(y)D_v(x)D_v(y)dxdy.$$

For the case $r(x, y)=\exp(-k|x-y|/|v-v_0|)$, and constant D_v, the author gets $D_I^2=2D_v^2(v-v_0)^2/k$, and regards D_I as negligible, whence the claim that integration suppresses the random errors. The application of this theorem to the variation of the output integral results in a definitive formula for the error of the integrator and specific suggestions for its reduction (say, elimination of deadlocks).

Bold generalizations follow a description of the Bush-type analyzer. They culminate in a statistical error theorem for systems of differential equations: for a system of the type $\phi_k(y^{(n)}, \cdots, y', y, \eta, \cdots)=0$, involving a number of given η-functions, the mean value of each y (neglecting second order quantities) is the integral of this system, with the η's and the initial values replaced by their mean values; and the dispersion D_y of η (neglecting second order terms again) is given by $D_y^2=\sum(\partial y/\partial\eta_i)_0^2 D_{\eta_i}(\Delta x)^2$, where the subscript 0 indicates the mean value, and η_i the mean value of η in the ith interval. If the ϕ_k's are linear in the $y^{(i)}$'s and η's, these relations hold rigorously. The same claim that D_y is negligible is made.

A. W. Wundheiler.

Bruevič, N. G. On the accuracy of the fundamental formula of the theory of the errors of a mechanism. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1944**, 545–558 (1944). (Russian)

The author nibbles at the problem of the error of the conventional first order error theory of linkage mechanisms. If q_s are the independent dimensions and position coordinates of an ideal linkage M_0, the position of a link of the real mechanism M can be represented by $\phi(q_s+\Delta q_s)$, and the error $\delta\phi$ of the linear error is given by the tail of the Taylor expansion of the last expression in powers of Δq_s, starting with the second order terms. The author writes $\delta\phi=\phi_0''\Delta q^2+\phi_0'''\Delta q^3+\cdots$, q being a singled out parameter. Here $\phi^{(n)}=(d^n\phi/dt^n)/q^n$, t is the time, and the subscript 0 refers to the ideal values of the parameters. The $\phi^{(n)}$ are given an easy kinematic interpretation allowing one to simplify their computation. Obvious upper bounds of $\delta\phi$ are stated, the expressions for mean $\delta\phi$ cited. One example is worked out in detail. The paper is needlessly elaborate.

A. W. Wundheiler (Chicago, Ill.).

Rapoport, I. M. On differentiation with calculating apparatus. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 1521–1542 (1947). (Russian)

This paper is concerned with a mathematical discussion of the problem of physically measuring d^jx/dt^j, $j=1, \cdots, n$, when $x(t)$ is known to satisfy an nth order linear differential equation with constant coefficients having right hand side constant, assuming that x may enter the differentiating apparatus with error $|\delta x(t)|\leqq\epsilon$. The constants a_{ij} in the system

$$(1) \qquad dx_i/dt+\sum_{j=1}^{n}a_{ij}(x_j-x)=0, \qquad i=1, \cdots, n,$$

and a_j are explicitly determined so that the quantity $y_j=a_j(x-x_j)$ approaches the jth derivative of x with increasing t. It is shown that

$$(2) \qquad y_j(t)=\int_0^\infty \frac{d}{dt}x(t-\tau)p_j(\tau)d\tau,$$

where $p_1(t)$ is a linear combination of exponentials in t with coefficients depending on $k_1, \cdots, k_n$, the roots of the auxiliary equation corresponding to a system obtained from (1) by change of variables, and $p_j(t)$ is a linear combination of derivatives of $p_1(t)$. Thus it appears that $y_j(t)$ is a mean value of the derivative calculated with weight $p_j(t)$. The representation (2) is convenient for discussion of the error in y_j due to $\delta x(t)$; it is found to be less than ϵ times a function of the k_i determined by $|p_j(0)|+\int_0^\infty|dp_j(t)|$. Hence the desired derivatives can be continuously measured with a degree of exactness corresponding to that of the given function $x(t)$ no matter how the error in x varies so long as it remains less than ϵ. This is not necessarily the case with the electrical tachometer which gives a voltage proportional to an instantaneous velocity of rotation.

The introductory section of the paper is devoted to the special case when dx/dt is constant. In this connection the author mentions two physical realizations of (1) for $n=1$: (a) the well-known mechanical device consisting of a Thomson type integrator operated by a constant speed motor and a differential, discussed with various modifications by H. S. Hele Shaw [Philos. Trans. Roy. Soc. London **176**, 367–402 (1885)] and (b) an electrical circuit similar to that discussed by F. J. Murray [The Theory of Mathematical Machines, King's Crown Press, New York, 1947, p. 42; these Rev. **9**, 103]. The closing section of the paper gives the detailed application of the general discussion to the case when d^2x/dt^2 is constant.

R. Church (Annapolis, Md.).

Rojansky, V. Gyrograms for simple harmonic systems subjected to external forces. J. Appl. Phys. **19**, 297–301 (1948).

The author describes a graphical solution of the linear oscillator equation (1) $m\ddot{x}+kx=f(t)$ with an "external force" $f(t)$ which is a step function in time. Representing the displacement x and the "velocity" $u=\dot{x}(m/k)^{\frac{1}{2}}$ as coordinates in an (x, u)-plane the solutions of (1) are circular

segments which are pieced together graphically. The method, whilst of instructional value, is likely to be more tedious than the more accurate use of sine tables. *H. O. Hartley.*

Horváth, J. I. An asymptotic solution of the fundamental equation of the statistical atom theory. Nature **161**, 26–27 (1948).

The fundamental equation of Thomas-Fermi may be written in the form (1) $Y''=x\{(Y/x)^{\frac{1}{2}}+\beta\}^3$ and is to be solved under the conditions $Y(0)=1$ and $Y(x_0)-x_0Y(x_0)=(Z-N)/Z$ (Z, N, x_0 denote the atomic number, the number of electrons and the atomic radius; β is a constant depending on Z). In the case $\beta=0$ (1) has the particular solution $Y=144/x^3$. In the general case $\beta\neq 0$ a solution is sought in the form $Y=(144/x^3)v(x)$. For $v(x)$ the author obtains the rapidly converging series

$$v(x)=1+c_1x^2+c_2x^4+c_3x^6+\cdots,$$

$$c_1=-\frac{3}{16}\beta,\ c_2=\frac{79}{9216}\beta^2,\ c_3=\frac{1584}{63700992}\beta^3.$$

This solution does not satisfy the condition $Y(0)=1$. The difficulty is avoided by using an artifice of Sommerfeld [Z. Physik **78**, 283–308 (1932)]: $x\to(x^2+a^2)$. This function

$$Y\sim 144(x^2+a^2)^{-3}\{1+c_1(x^2+a^2)+c_2(x^2+a^2)^2+c_3(x^2+a^2)^3+\cdots\}$$

is asymptotically equal to the first, and the parameter a may be chosen so that the condition $Y(0)=1$ is satisfied. [It may be remarked that this solution does not satisfy (1); moreover the second boundary condition is not considered.] The author's solution is said to yield a very good approximation, especially in the exterior parts of the atoms. *S. C. van Veen* (Delft).

Unger, Heinz. Zur numerischen Behandlung von Anfangswertproblemen bei gewöhnlichen linearen Differentialgleichungen 2. Ordnung. Z. Angew. Math. Mech. **25/27**, 135–136 (1947).

The author is concerned with approximate numerical methods of solving the initial value problem of the ordinary second order differential equation (1) $\ddot{v}+d(t)\dot{v}+c(t)v=0$. Transformations (some of which are well known) of both dependent and independent variable to simplify (1) are summarily discussed. It is suggested in particular that the reduced form (2) $w''+f(x)w=0$ be solved approximately by splitting the variate range of x into intervals such that over each interval $f(x)$ can be approximately represented by a polynomial $\rho(x)$ of order 0, 1 or 2 in x, so that the solution of (2) is given by trigonometrical functions, Airy integrals, Weber functions or other higher mathematical functions. The error committed by approximating to $f(x)$ by $\rho(x)$ is gauged in the usual way with the help of the integral equation associated with (2). The cumulative error in the initial conditions from interval to interval is not discussed, however. The method will become tedious (if not unmanageable) if $f(x)$ and its derivatives are large. A priori bounds for the solution of (2), depending on the coefficients only and not on the knowledge of the approximate solutions, are derived. *H. O. Hartley* (London).

Pfeiffer, F. Über die Differentialgleichung der transversalen Stabschwingungen. Z. Angew. Math. Mech. **25/27**, 83–91 (1947). (German. Russian summary)

The author is concerned with solving the equation

$$(1)\qquad a\frac{\partial^4 z}{\partial x^4}-b\frac{\partial^4 z}{\partial x^2\partial y^2}+c\frac{\partial^4 z}{\partial y^4}+d\frac{\partial^2 z}{\partial y^2}=0$$

arising in the propagation of vibrations in a beam where the constants a to d are composed of the physical constants of the beam. Using the theory of characteristics the author suggests various methods of reducing the solution to that of a system of simultaneous ordinary differential equations. The method advocated for numerical treatment is that of reducing (1), first to a system of two second-order hyperbolic partials, which are then transformed to standard form by introducing the characteristic parameters as independent variables. The characteristics of these two simultaneous partials require the solution of a system of 8 simultaneous, first order, ordinary differential equations of a simple type. A numerical method is outlined (not executed), being a step by step integration with a low order finite difference formula and iterative improvement. Since interpolation is required at each stage the method appears to be more laborious than finite difference technique or relaxation applied directly to the original equation. The author's theory provides estimates for the propagation of discontinuities. *H. O. Hartley* (London).

Gray, Marion C., and Schelkunoff, S. A. The approximate solution of linear differential equations. Bell System Tech. J. **27**, 350–364 (1948).

Following a previous paper [S. A. Schelkunoff, Quart. Appl. Math. **3**, 348–355 (1946); these Rev. **7**, 300] the authors deal with the initial value problem of the second order equation $y''=F(x)y$ which they solve for $a\leqq x\leqq b$, and given $y(a)$ and $y'(a)$. They replace the equation by (1) $y''+\beta^2y=(\beta^2+F(x))y=f(x)y$ (say), where β is a suitably chosen average of $\sqrt{F}$. Starting from the sinoidal solution of the homogeneous equation $y''+\beta^2y=0$ they select the particular solution $W_0(x)$ satisfying the initial conditions at $x=a$ and then solve the inhomogeneous equation (1) by the usual iterative sequence $y(x)=\sum_{n=0}^{\infty}W_n(x)$, where the $W_n(x)$ are recurrently defined by

$$(2)\qquad W_n(x)=\beta^{-1}\int_a^x f(u)W_{n-1}(u)\sin\beta(x-u)du.$$

Numerical examples are given showing the rapidity of the convergence compared with Picard's classical method and mainly concerned with the Airy integral equation $y''=xy$. No mention is made of the recently published tables [J. C. P. Miller, The Airy Integral, British Association, Mathematical Tables, Part-volume B, 1946; these Rev. **8**, 353] and the various methods of solution given in the introduction of these tables. The method of quadrature for (2) is not stated but with numerical quadrature difficulties will arise if F (and hence β) as well as F' are large. *H. O. Hartley* (London).

Kron, Gabriel. Electric circuit models of partial differential equations. Elec. Engrg. **67**, 672–684 (1948).

A lecture summarizing recent work of the author.

Kantorovič, L. V. On Newton's method for functional equations. Doklady Akad. Nauk SSSR (N.S.) **59**, 1237–1240 (1948). (Russian)

Let X be a Banach space, $P(x)$ an operator transforming space X into space Y of the same type. Consider the equation (1) $P(x)=0$, where P is capable of being differentiated twice in the sense of Fréchet. Let the following conditions be satisfied: (i) the inverse operator $\Gamma_0=[P'(x_0)]^{-1}$ exists such that $\|\Gamma_0\|\leqq B_0$; (ii) the element x_0 of the approximation satisfies (1); (iii) $\|\Gamma_0P(x_0)\|\leqq\eta_0$, $\|P''(x)\|\leqq K$; (iv) $B_0\eta_0K=h_0\leqq\frac{1}{2}$;

then (1) possesses a solution x^* which can be obtained by Newton's algorithm:

$$\|x_0-x^*\|\leqq N(h_0)\eta_0=\{1-(1-2h_0)^{\frac{1}{2}}\}h_0^{-1}\eta_0,$$

the rapidity of the convergence being

$$\|x_n-x^*\|\leqq 2^{-n+1}(2h_0)^{2^n-1}\eta_0$$

and the solution unique inside the region

$$\|x-x_0\|<L(h_0)\eta_0=\{1+(1-2h_0)^{\frac{1}{2}}\}h_0^{-1}\eta_0 \tag{2}$$

(if $h_0=\frac{1}{2}$, the sign $<$ in (2) must be replaced by $\leqq$, the condition $\|P''(x)\|\leqq K$ remaining fulfilled in the region). Modifications of Newton's method as follow can also be employed: $x_{n+1}=x_n-\Gamma_0P(x_n)$ (condition of convergence: $h_0=\sqrt{2}-1$) or $x_{n+1}=x_n-Q[P'(x_n)]^{-1}P(x_n)$, $0<Q<1$; $h_0\leqq\frac{1}{2}$ (method of Grave).

For system of algebraic or transcendental equations, consider an m-dimensional space X with the norm $\|x\|=\sup(|\xi_1|,\cdots,|\xi_m|)$. Given the system

$$f_i(\xi_1,\cdots,\xi_m)=0,\qquad i=1,\cdots,m, \tag{3}$$

suppose that $|f_i(\xi_1^0,\cdots,\xi_m^0)|\leqq\eta$ $(i=1,\cdots,m)$ and that the m by m matrix $\|\partial f_i/\partial f_k\|$ possesses a nonvanishing determinant Δ. Define $B=\Delta^{-1}\max\sum_k A_{ik}$, where A_{ik} are the minors of Δ. Suppose also that $|\partial^2 f_i/\partial\xi_i\partial\xi_j|\leqq L$ in the region under consideration; then, provided that $h=B^2n^2\eta L\leqq\frac{1}{2}$, the system (3) possesses a solution and this can be found by Newton's method.

For integral equations consider, in the space X of measurable bounded functions,

$$x(s)=\int_0^1 K(s,t,x(t))dt. \tag{4}$$

This equation possesses a solution, provided

$$\left|x_0(s)-\int_0^1 K(s,t,x_0(t))dt\right|\leqq\eta,$$

and the kernel $K_x'(s,t,x_0(t))=K(s,t)$ possesses a resolvent kernel $\Gamma(s,t)$ such that $\int_0^1\Gamma(s,t)dt\leqq B$, $0\leqq s\leqq 1$, $K''(s,t,u)\leqq K$ and $(B+1)^2\eta K\leqq\frac{1}{2}$. D. M. Zagadskiĭ gave a similar theorem [same vol., 1041–1044 (1948); these Rev. **9**, 443] with the number $\frac{1}{10}$ instead of $\frac{1}{2}$ in the last condition.

M. Daniloff (Cambridge, Mass.).

Gradštein, I. S. An apparatus for drawing the graph of a function of a function. Trav. Inst. Math. Stekloff **20**, 129–130 (1947). (Russian)

The device consists of a frame capable of movement to the left and right on which a plate carrying a template for $f(x)$ can move also to the left and right. One arm of an L-shaped lever which is pivoted to the frame positions the plate relative to the frame by means of a sliding connection; the other arm carries a tracing point which slides on it and on a rod attached to the frame. When the graph of $g(x)$ is traced, a follower of the template profile (moving on the rod on which the tracing point slides) describes $f(kg(x))$ relative to the plane on which $g(x)$ is drawn. Essentially the same linkage occurs in the Nyström-Stieltjes planimeter described by E. Laurila [Soc. Sci. Fenn. Comment. Phys.-Math. **10**, no. 7 (1940); these Rev. **2**, 240] where omission of the integrating device, interchange of dependent and independent variables and slight alteration of the sliding connections would produce the present instrument. No details of construction or indication that the device has been built are given. *R. Church* (Annapolis, Md.).

RELATIVITY

Karmarkar, K. R. Gravitational metrics of spherical symmetry and class one. Proc. Indian Acad. Sci., Sect. A. **27**, 56–60 (1948).

It is shown that every spherically symmetric gravitational field is of class two, i.e., it can be immersed in a flat 6-space. Conditions are obtained for a spherically symmetric metric to be of class one. Perfect fluid distributions are discussed which satisfy these conditions. *A. Schild.*

Buchdahl, H. A. A special class of solutions of the equations of the gravitational field arising from certain gauge-invariant action principles. Proc. Nat. Acad. Sci. U. S. A. **34**, 66–68 (1948).

The following theorem is proved. In a region free from radiation the field equations arising from a gauge invariant action principle, in which the Lagrangian is composed of the components of the contracted curvature tensor and the components of the metrical tensor, are satisfied by any solution of the equations $G_{ij}=ag_{ij}$ where a is an arbitrary constant. *M. Wyman* (Edmonton, Alta.).

Fourès-Bruhat, Yvonne, et Lichnerowicz, André. Sur un théorème global de réduction des ds^2 statiques généraux d'Einstein. C. R. Acad. Sci. Paris **226**, 775–777 (1948).

Let (*) $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$ be the metric of a universe corresponding to distributions of mass at finite distance, and satisfying the interior and exterior Einstein equations, respectively, in regions containing matter and free from matter. Suppose also that the universe (*) satisfies the following conditions: (a) the ds^2 in all regions containing matter, and hence in the region exterior thereto, are static of the most general type, the $g_{\alpha\beta}$ being independent of the temporal variable x^4; (b) the metric (*) is asymptotically Euclidean at infinity. Then the authors prove that ds^2 is necessarily of Levi-Civita orthogonal type throughout the whole universe under either of the following hypotheses: (i) the lines of flow inside matter (treated as a perfect fluid) coincide with the time-curves; (ii) the interior ds^2 are all static and orthogonal. They conclude with the remark that this theorem allows, for example, a considerable reduction in the number of postulates, a priori independent, that lead to the formation of the interior and exterior Schwarzschild metrics. *H. S. Ruse* (Leeds).

Jordan, P. Erweiterung der projektiven Relativitätstheorie. Ann. Physik (6) **1**, 219–228 (1947).

Projective relativity theory is based on the invariant $J=X_\nu X^\nu$ $(\nu=1,\cdots,5)$ and it is usually assumed arbitrarily that this invariant is a constant. If this condition is relaxed it is shown that J is proportional to the gravitational constant κ which then appears as a variable in a manner which it is suggested corresponds to the supposition of Dirac [Proc. Roy. Soc. London. Ser. A. **165**, 199–208 (1938)]. Here κ is in fact capable of even greater freedom of variation, for it is a space-time scalar field quantity.

H. C. Corben (Pittsburgh, Pa.).

Van Bergen, F. Forme invariante des équations de Dirac. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 638–651 (1947).

It is known that an arbitrary Lorentz transformation L_ν^μ may be written as $L_\nu^\mu=(s\delta_\nu^\mu+s^\alpha Z_{\alpha\nu}^\mu)(s\delta_\rho^\nu+s^\alpha\bar{Z}_{\alpha\rho}^\nu)$, where

z, z^α are four complex numbers subject to a normalization condition $z^2+\sum_{\alpha=1}^{3}(z^\alpha)^2=1$ and $Z_\alpha^{\mu\nu}=Z_{\alpha\rho}^{\mu}g^{\rho\nu}$ are a set of three anti-symmetric self-dual tensors. The author confuses the four numbers (z, z^α) with the components of a quaternary spinor and attempts to identify them with the Dirac wave functions. It is known that these four numbers are simply related to a two-index binary spinor which may be constructed from any pair of simple binary spinors, in particular those satisfying the Dirac equation. When this is done one obtains tensor equations equivalent to the Dirac ones. *A. H. Taub* (Princeton, N. J.).

Gosselin, Jacques. Sur le déplacement des spectres des nébuleuses vers le rouge et l'évolution de l'univers. C. R. Acad. Sci. Paris **226**, 228–230 (1948).

The author considers the red displacement in a universe whose line element is spherically symmetric and such that its gravitational potentials are functions of time.

M. Wyman (Edmonton, Alta.).

Ives, Herbert E. The behavior of an interferometer in a gravitational field. II. Application to a planetary orbit. J. Opt. Soc. Amer. **38**, 413–416 (1948).

This paper continues the investigation of the author's formulation of the principle of general relativity [same J. **29**, 183–187 (1939)]. The equations of planetary orbits are studied and are shown to give the correct advance of perihelion. *A. Schild* (Toronto, Ont.).

Takeno, Hyôitirô. Cosmology and conformally flat space. II. J. Sci. Hirosima Univ. Ser. A. **11**, 201–230 (1942).

[For part I see same J. Ser. A. **10**, 173–214 (1940); these Rev. **2**, 208.] This paper is concerned with various forms for the line element of a cosmological space. These are derived by transformations of coordinates from the line element

$$ds^2=-F(r,t)(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2)+dt^2,$$

where $F(r,t)=e^{2v(t)}/[1+\frac{1}{4}r^2/R^2]^2$. In the discussion use is made of the existence of a scalar v such that $v_{,ij}=\beta g_{ij}$, where the comma denotes the covariant derivative.

A. H. Taub (Princeton, N. J.).

Pasturaud, Marie-Thérèse. Sur l'existence d'échelles de temps privilégiées en physique théorique. Cahiers de Physique nos. **31–32**, 92–120 (1948).

Concepts and postulates are formulated to agree as closely as possible with our intuitive idea of time. Most of the discussion is in relation to a single observer and the primary concepts are of an event perceived by the observer, the simultaneity of events, the "memory" of an event, the possibility (in the future) of an event, and the true future. Postulates are given which lead to the familiar order relations between events. Clocks are introduced, and an instant is defined as a number attached to an event. The existence of privileged time scales is deduced from the existence of canonical forms for certain operators of Hamiltonian type. For more than one observer the discussion follows very closely the work of Milne and Whitrow [cf. Z. Astrophys. **15**, 263–298 (1938)], particularly in relation to the t- and τ-scales of time. *A. G. Walker* (Sheffield).

Lewis, T. Kinematical relativity. Philos. Mag. (7) **38**, 602–606 (1947).

A reply to W. Band's criticism [Philos. Mag. (7) **37**, 551–563 (1946); these Rev. **8**, 608] of Milne's kinematical relativity theory. *A. Schild* (Toronto, Ont.).

Causse, Maurice. Sur la théorie cosmologique de E. A. Milne. Démonstration de la relation $G(\xi)=-1$ pour les systèmes cinématiques simples. C. R. Acad. Sci. Paris **226**, 876–878 (1948).

Milne's cosmological theory is briefly reviewed and his deduction of a law of gravitational attraction is criticised.

A. Schild (Toronto, Ont.).

Gregory, Christopher. A note on quantized space. Physical Rev. (2) **73**, 806–807 (1948).

Snyder [same Rev. (2) **71**, 38–41 (1947); these Rev. **8**, 412] has shown that a Lorentz invariant space-time is not necessarily a continuum. The infinitesimal elements of the group under which $-\eta^2=\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2-\eta_4^2$ is invariant are at least 15 in number: Snyder used 10. For $\eta^2>0$ five infinitesimal elements B_0, B_j $(j=1, 2, 3, 4)$, and another set $\bar{B}_0$, $\bar{B}_j$ for $\eta^2<0$, are given whose ten commutators generate the elements used by Snyder, apart from a numerical factor. The commutators of the fifteen elements give nothing essentially new. The spectra of B_0, $\bar{B}_j$ are discrete, those of $\bar{B}_0$, B_j continuous. The full physical significance of these has not been found. *C. Strachan* (Aberdeen).

Ditchburn

Ditchburn, R. W. Phase-velocity and group-velocity in relativistic optics. Rev. Optique **27**, 4–14 (1948).

The author examines the hypothesis that light propagation in vacuo is dispersive. It follows that the limiting velocity c_l of signals, the phase velocity c_p and group velocity c_g of light are all different. The Doppler effect, aberration, and the experiments of Fizeau and Michelson-Morley are discussed. Although the theory is said to be relativistic, it admits the existence of an absolute frame of reference.

A. Schild (Toronto, Ont.).

Dreyfus-Graf, J. Les formules de l'effet Doppler dans la théorie ellipsoïdale de la relativité restreinte (erreur des formules d'Einstein). Helvetica Phys. Acta **21**, 87–92 (1948).

The author computes the Doppler effect on his "ellipsoidal" theory of light propagation [same Acta **19**, 399–404 (1946)] and arrives at a formula differing from the usual relativistic one. *A. Schild* (Toronto, Ont.).

MECHANICS

Pailloux, Henri. Petits mouvements verticaux périodiques d'un câble souple, inextensible, pesant, dont les extrémités sont fixes. C. R. Acad. Sci. Paris **226**, 1242–1244 (1948).

Consider a cable whose static equilibrium position is a catenary. This note is concerned with the small periodic oscillations of the cable in the plane of the equilibrium curve. The appropriate differential equation is derived and an approximate solution is offered for the case where the cable is very tight. *G. F. Carrier* (Providence, R. I.).

Hostinský, Bohumil. Sur le mouvement asymptotique d'un pendule après un très grand nombre de chocs successifs du pendule avec des corpuscules. C. R. Acad. Sci. Paris **226**, 990–992 (1948).

Suppose that at each of the instants $t=0, \tau, 2\tau, 3\tau, \cdots$ a pendulum (or other dynamical system having one degree of freedom) is struck by a particle, the particles having equal masses and equal velocities before collision with the pendulum. The author calculates the asymptotic state of

motion of the pendulum, which is approached after a large number of collisions. He suggests that the results may be applicable to some problems in atomic physics.

L. A. MacColl (New York, N. Y.).

Stihi, E. E. Sur un problème de balistique. Acad. Roum. Bull. Sect. Sci. **24**, 417–420 (1943).

The author treats the ballistic problem of the descending branch of a trajectory, with drag coefficient following the square law. [He ignores temperature effects on sound velocity.] Using $\eta=((H-Z)/H)^{\frac{1}{2}}$, where H is the initial altitude and Z the altitude at time t, he proposes a power series solution in η for the slope (tangent θ), and points out its convergence in the case of bombardment by an airplane for steady altitude of flight. *A. A. Bennett.*

Kasner, Edward, and De Cicco, John. Physical curves in generalized fields of force. Proc. Nat. Acad. Sci. U. S. A. **34**, 169–172 (1948).

In the authors' terminology a generalized field of force is a field of force such that the force acting on a particle depends upon the position of the particle and upon the direction of motion (without regard to sense) of the particle. Also, a system S_k of curves consists of curves along which constrained motion is possible with the pressure P and the normal component of force N related by the equation $P=kN$, where k is a constant. In this note the case of a particle moving in a plane in a generalized field of force is considered, and some of the properties of the resulting systems S_k are stated. The most notable result is a generalization of Kasner's theorem to the effect that if a particle starts from rest in a positional field of force, the initial curvature of the trajectory is one-third of the curvature, at the initial point, of the line of force passing through that point. *L. A. MacColl* (New York, N. Y.).

Mendes, Marcel. Sur les transformations canoniques. C. R. Acad. Sci. Paris **226**, 1240–1242 (1948).

The author obtains necessary and sufficient conditions that a transformation carry the canonical system,

$$dp_i/dt=-\partial H/\partial q_i, \quad dq_i/dt=\partial H/\partial p_i, \qquad i=1, \cdots, n,$$

into the system $dP_i/dt=-\partial F/\partial Q_i$, $dQ_i/dt=\partial F/\partial P_i$, with $kF(P, Q, t)=H(p, q, t)-\partial V(q, Q, t)/\partial t$, where k is a constant. The results are simple generalizations of those obtained in the well-known special case when $k=1$.

D. C. Lewis (Baltimore, Md.).

Castoldi, L. Il principio di Hamilton per sistemi dinamici a vincoli anolonomi generali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 329–333 (1947).

Hamilton's principle in the form

$$\int_{t_0}^{t_1}\left(\delta T+\sum_{k=1}^{n}Q_k\delta q_k\right)dt=0$$

is extended to the case of nonholonomic systems in which the constraint equations $f_r(q, \dot{q}, t)=0$, $r=1, \cdots, l<n$, are not necessarily linear in the $\dot{q}$'s. The "virtual variations" δq_i are assumed to satisfy $\sum_{k=1}^{n}(\partial f_r/\partial \dot{q}_k)\delta q_k=0$.

D. C. Lewis (Baltimore, Md.).

Nikitin, A. K. On certain properties of the trajectories of a conservative system. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 23–28 (1948). (Russian)

Elementary results concerning the behavior of the characteristics of the Hamilton-Jacobi equation in the neighborhood of a point. *G. A. Hedlund* (New Haven, Conn.).

Hydrodynamics, Aerodynamics, Acoustics

***Prandtl, Ludwig. Führer durch die Strömungslehre.** Friedr. Vieweg & Sohn, Braunschweig, 1942; J. W. Edwards, Ann Arbor, Mich., 1947. xi+383 pp. $4.80.

This is an enlarged third edition of the author's Abriss der Strömungslehre [Vieweg, Braunschweig, 1931]. While written primarily for engineers and physicists it should prove of great value for applied mathematicians since it covers practically all fields of applied fluid mechanics, reporting exhaustively on the main physical facts and experimental techniques and giving many typical experimental results. Mathematical methods are, of course, described too, though only relatively simple methods leading immediately to practically important results are stressed. The book is divided into five chapters: I. Properties of liquids and gases. Theory of equilibrium states. II. Kinematics of fluids. Dynamics of perfect fluids. III. Motion of viscous fluids. Turbulence. Resistance. Technical applications. IV. Gas dynamics. V. Special problems (Interaction of different states; flow of heavy liquids; heat transfer in fluids).

L. Bers (Syracuse, N. Y.).

Allen, D. N. de G. Analytical solution of a four-cusped wake. Nature **160**, 509 (1947).

Let w be the complex velocity potential of a plane incompressible potential flow, and $\zeta=q^{-1}e^{i\theta}$, where q is the flow speed and θ the inclination of the flow direction. Then the flow defined by

$$w=(d/\pi)\log((1+t)/(1-t)), \quad t=\mathrm{sn}((4Ki/\pi)\log\zeta-K),$$

d, K constants, is that of two equal and opposite streams of width $2d$ approaching one another and bifurcating symmetrically along a four-cusped closed curve on which $q=$constant, an explicit example of a finite closed wake with cusped bounding streamlines. *D. Gilbarg.*

Lighthill, M. J. Note on the ultimate form of rotary fluid motion inside a cylinder. Quart. J. Math., Oxford Ser. **19**, 65–66 (1948).

The author solves the equation $v_t=\nu(v_{rr}+r^{-1}v_r-r^{-2}v)$, under the boundary conditions $v(r, 0)=v_0(r)$, $v(0, t)=v(a, t)=0$, by using a Fourier-Bessel series. For large t the first term dominates, so that the distribution of the velocity v along a radius r is like that of the Bessel function J_1 between 0 and its first positive zero. The motion will persist longest if $v_0(r)$ is chosen to make the dominant term as large as possible; "so, to any person to whose advantage it may be that a rotary fluid motion in a cylinder, produced by stirring, may persist as long as possible, the following advice is tendered: 'stir in a circle, concentric with the cylinder, of about 0.63 times its radius.' " Here 0.63 is an approximation to the ratio of the least positive zeros of J_0 and J_1.

R. P. Boas, Jr. (Providence, R. I.).

Bouligand, Georges. Un cas typique d'entraînement d'un liquide visqueux. C. R. Acad. Sci. Paris **226**, 1571–1573 (1948).

Vinogradov, Yu. P., and Kufarev, P. P. On a problem of filtration. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 181–198 (1948). (Russian)

The following problem is considered. In an incompressible fluid, which fills some two-dimensional region G_t that varies with time t, there exists at a point $z=0$ an outflow of magnitude $2\pi g(t)$ which changes with time. The velocity poten-

tial is assumed to be constant on the boundary C_t of G_t during some interval of time. It is required to determine the nature of the motion and the shape of the region G_t if the region G_0 occupied by the fluid at time $t=0$ is given. On the basis of the results of P. J. Poloubarinova-Kochina [Appl. Math. Mech. [Akad. Nauk. SSSR. Prikl. Mat. Mech.] 9, 79–90 (1945); C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 250–254 (1945); these Rev. 7, 95, 140], and L. A. Galin [C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 246–249 (1945); these Rev. 7, 229] this problem is reduced to the solution of a system of integro-differential equations. A study of these equations permits the authors to draw certain conclusions on the existence and nature of the solution of a boundary value problem to which the present problem had been reduced by Poloubarinova-Kochina and to develop two methods for the solution of the problem.

H. P. Thielman (Ames, Iowa).

Kalinin, N. K. On the instability of filtration in the case of a drain in a water-permeable layer of finite depth. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 199–206 (1948). (Russian)

The same problem on the lowering of the water table as that of an earlier paper [N. K. Kalinin and P. J. Poloubarinova-Kochina, Appl. Math. Mech. [Akad. Nauk. SSSR. Prikl. Mat. Mech.] 11, 231–236 (1947); these Rev. 9, 118] is considered here with the only difference that the drain is now not located in a semi-infinite plane, but at the bottom of a layer of unit thickness. *H. P. Thielman.*

Vedernikov, V. V. On the theory of drainage. Doklady Akad. Nauk SSSR (N.S.) 59, 1069–1072 (1948). (Russian)

The author treats a sufficiently general problem of draining a layer of soil, when the cause of waterlogging is not only atmospheric precipitation or irrigation from above, but also infiltration of water from a lower layer of soil due to hydrostatic pressure and capillarity. The method of attack has been previously described by the author [C. R. Acad. Sci. Paris 202, 1155–1157, 1387 (1936)].

A. W. Boldyreff (Albuquerque, N. M.).

von Kármán, Theodore. Effect of compressibility in aerodynamics. Ciencia y Técnica 110, 251–278, 305–320 (1948). (Spanish)

Translation of J. Aeronaut. Sci. 8, 337–356 (1941); these Rev. 3, 220.

Kiebel, I. A. Exact solutions of equations of gas dynamics. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 193–198 (1947). (Russian. English summary)

The author constructs exact solutions of the two-dimensional problem of gas dynamics with nonvanishing vorticity. The solutions are obtained by taking as independent variables the coordinate x and the stream-function ψ and by setting p = pressure = $\partial\chi/\partial x$, v = y-component of the velocity = $\partial\chi/\partial\psi$, and by assuming that the function χ is of the form $\chi=-H(\psi)x^{-(\gamma-1)/(\gamma+1)}$, γ = ratio of specific heats. The function H must satisfy a certain ordinary differential equation. The solutions obtained exhibit both subsonic and supersonic speeds. A similar procedure goes through for a flow with rotational symmetry. [R. C. Prim states in a letter to the reviewer that according to his computations Kiebel's flows involve negative pressures and densities.]

L. Bers (Syracuse, N. Y.).

Ringleb, F. An approximation procedure for determining the pressure distribution in an adiabatic gas flow. The Graduate Division of Applied Mathematics, Brown University. Translation no. A9-T-5. i+38 pp. (1948).

[Translation of Messerschmidt Report no. Me/Ge/Re/244 (1940).] It is well known that the stream function ψ when expressed in terms of the magnitude of velocity w and the inclination θ of the velocity vector can be written as

$$\psi=\sum_n P_n(w)\{A_n\cos n\theta+B_n\sin n\theta\},$$

where A_n and B_n are constants and $P_n(w)$ are functions connected with the hypergeometric functions. For small values of w, $P_n(w)$ becomes w^n. The author proposes to replace $P_n(w)$ as an approximation by

$$P_n(w)=w^n\{1-\tfrac{1}{4}(w/c_0)^2+\tfrac{1}{40}(w/c_0)^4\}^n,$$

where c_0 is the stagnation sound velocity. Therefore one can construct an approximate compressible solution in the hodograph plane by simply replacing the velocity w_i in an incompressible solution by a new velocity scale W,

$$W=w[1-\tfrac{1}{4}(w/c_0)^2+\tfrac{1}{40}(w/c_0)^4].$$

The coordinates (x, y) in the physical plane can be obtained by numerical quadratures.

The proposed solution, being actually a solution of an elliptic equation, can be accurate only for purely subsonic flows. But then the velocity scale W is different from the Kármán-Tsien scale [Tsien, J. Aeronaut. Sci. 6, 399–407 (1939)] by less than 0.5%. The Kármán-Tsien solution has, however, the advantage of being able to express the physical coordinates in closed forms. The author's application of his approximate solution to cases where limiting lines occur (locally supersonic flows) is unacceptable.

H. S. Tsien (Cambridge, Mass.).

Simonov, L. A. Calculation of an aerofoil in a flow and plotting of an aerofoil according to a distribution of velocities over its surface. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 69–84 (1947). (Russian. English summary)

Using the hodograph method, the author describes a procedure for the determination of a flow pattern around a profile approximating the given one. By the transformation $\zeta=\xi(z)$, $\zeta=|\zeta|\exp(i\theta)$, $z=x+iy$, the exterior of the profile is mapped into the exterior of the unit circle. The author considers the flow in the Λ-plane, $\Lambda=1/V$, where V is the conjugate to the velocity vector. On the contour we have $\Lambda(\theta)=dz/d\theta=dx/d\theta+idy/d\theta$, and the functions $\lambda_x=dx/d\theta$, $\lambda_y=dy/d\theta$ (being conjugate harmonic functions) can be expressed in terms of each other; $x(\theta)$, $y(\theta)$ and the speed $v(\theta)$ on the boundary of the profile can be expressed in terms of $\lambda_x(\theta)$ or $\lambda_y(\theta)$. The exact expressions (involving integrals) are approximated by finite sums. Tables of the values of $\lambda_x(\theta)$, $\lambda_y(\theta)$, $z(\theta)$ for a number of profiles are given. The problem of determining the airfoil from the given velocity components and the inverse problem are solved by making the necessary corrections for an auxiliary profile for which the tables are available. *S. Bergman.*

Kochin, N. E. Calculation of the hydrodynamic characteristics of a large interval grid. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 85–96 (1947). (Russian. English summary)

This paper contains a new derivation of the author's approximate theory of an incompressible two-dimensional potential flow past a straight cascade of profiles with a

large gap-chord ratio. The following notations are used: t is the distance between two congruent points on adjacent profiles, β is the angle between the y-axis and the axis of the cascade, $\tau = te^{-i\beta}$, $f(z)$ is the complex potential, $w(z)$ is the conjugate complex velocity, $w_{\pm\infty}$ are the values of w far upstream and far downstream, $w_0 = \frac{1}{2}(w_{+\infty} + w_{-\infty})$, Γ is the circulation of the cascade-flow around one profile, Γ_0 is the circulation of a flow past a single profile (congruent to the cascade profile) for the free-stream velocity w_0, θ_0 is the angle of attack of the noncirculatory flow past a single profile, r is the conformal radius of a single profile, θ and l are the angle of inclination and length of a segment of a straight segment cascade "equivalent" to the given one.

The author first obtains for $w(z)$ the expression

$$w(z) = w_0 + (2\pi i)^{-1} \int_C w(\zeta) \coth \{\pi(z-\zeta)/\tau\} d\zeta,$$

where the integration is performed along a simple profile. He expands the integral in a power series in $1/\tau$ and retains only terms up to $1/\tau^4$. From this he concludes that for large enough values of $|\tau|$ the cascade-flow may be interpreted as the flow past a single profile whose conjugate complex velocity behaves at infinity as $w_0 + Ez$, where E is a certain constant. This flow can be computed using conformal mapping. Let $z = F(u) = u + k_0 + k_1/u + \cdots$, $k_1 = m^2 e^{2i\gamma}$, be the mapping of $|u| > r$ onto the domain exterior to a single profile. The following formulas are obtained:

$$\theta = \theta_0 - \delta, \quad \Gamma = 4\pi r k |w_0| \sin(\theta_0 - \alpha - \delta), \quad l = 4k'r,$$

where

$$k = 1 - \tfrac{1}{2}\pi^2 t^{-2}[m^2 \cos 2(\gamma+\beta) + 3r^2 \cos 2(\theta_0+\beta)],$$
$$k' = 1 - \tfrac{1}{2}\pi^2 t^{-2}[m^2 \cos 2(\gamma+\beta) - r^2 \cos 2(\theta_0+\beta)],$$
$$\delta = \tfrac{1}{2}\pi^2 t^{-2}[m^2 \sin 2(\gamma+\beta) - r^2 \sin 2(\theta_0+\beta)].$$

The paper concludes with an example and with a description of a graphical method for the approximate construction of the equivalent straight segment cascade. *L. Bers.*

Milne-Thomson, L. M. Applications of elliptic functions to wind tunnel interference. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 316–318 (1948).

The author considers a wind tunnel whose section is a given curve in the z-plane, the region interior being mapped within the unit circle of a Z plane by $z = f(Z)$. A general formula is then written for the additional upwash (interference velocity) at a wing of elliptic loading, in terms of $f(Z)$. The particular cases of elliptical and rectangular tunnels are treated; for these, the transformation appears in suitable Jacobian elliptic functions. For the elliptic tunnel, the formulae are worked out in some detail, assuming the wing to lie on the major axis. Finally, when the wing tips coincide with the foci, the formulae can be integrated in closed form, giving results in agreement with Glauert's [British Aero. Res. Comm. Rep. and Memo. no. 1470 (1932)]. *W. R. Sears* (Ithaca, N. Y.).

Perl, W., and Moses, H. E. Velocity distributions on symmetrical airfoils in closed tunnels by conformal mapping. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1642**, 58 pp. (1948).

Brun, Edmond, et Vasseur, Marcel. Étude de la couche limite autour d'un profil quelconque. C. R. Acad. Sci. Paris **226**, 1428–1430 (1948).

The authors announce an approximate method of computing both the velocity and temperature boundary layer of a profile with known pressure distribution. The method is based on a comparison of characteristic boundary layer parameters (e.g., the momentum thickness) between a profile with known boundary layer and the profiles for which the boundary layer is to be computed. For the known case the roof-profile is chosen. *H. W. Liepmann.*

Tournier, Marcel, et Bassière, Marc. Sur une solution des équations de la couche limite. C. R. Acad. Sci. Paris **226**, 1124–1127 (1948).

The one-dimensional nonsteady equations of motion have the well-known solution [Rayleigh]

$$U = U_0\{1 - \operatorname{erf}[\tfrac{1}{2}y(\nu t)^{-1}]\}$$

corresponding to the impulsive motion of a flat plate in a viscous fluid. The authors attempt to extend this solution to boundary layer flow. This is done in two ways: by introducing two constants to be fitted to Blasius' solution and by trying to find a solution of the boundary layer equation in terms of an incomplete Γ-function. The purpose of the investigation is to find functions which are simpler than Blasius' series. *H. W. Liepmann* (Pasadena, Calif.).

Agostini, Léon. Équation de la couche limite laminaire dans un convergent conique. C. R. Acad. Sci. Paris **226**, 1684–1685 (1948).

Nitzberg, Gerald E., and Crandall, Stewart. Some fundamental similarities between boundary-layer flow at transonic and low speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1623**, 30 pp. (1948).

Lewis, J. A. Boundary layer in compressible fluid. Prepared under the supervision of G. F. Carrier. Tech. Rep. no. **F-TR-1179-ND (GDAM A-9-M V)**. Headquarters Air Materiel Command, Wright Field, Dayton, Ohio. iv+66 pp. (1948).

The author sets out the equations of gas flow and of the compressible boundary layer and gives a connected summary of the work of Busemann [Z. Angew. Math. Mech. **15**, 23–25 (1935)], Kármán and Tsien [J. Aeronaut. Sci. **5**, 227–232 (1938)], Emmons and Brainerd [J. Appl. Mech. **8**, A-105–A-110 (1941); these Rev. **3**, 284] and Crocco [Monografie Scientifiche di Aeronautica, no. **3** (1946)] on the problem of the flat plate. There are appendices on the effects of radiation and on the boundary layer on bodies of revolution. *M. J. Lighthill* (Manchester).

Young, A. D. Note on the velocity and temperature distributions attained with suction on a flat plate of infinite extent in compressible flow. Quart. J. Mech. Appl. Math. **1**, 70–75 (1948).

Young, A. D. Note on the velocity and temperature distributions attained with suction on a flat plate of infinite extent in compressible flow. Coll. Aeronaut. Cranfield. Rep. no. **8**, 8 pp. (2 plates) (1947).

The boundary layer equations for the flat plate with suction in compressible flow are studied under the assumption that the Prandtl number is constant and the viscosity varies as a power of the absolute temperature. Final velocity and temperature profiles are obtained by establishing a correspondence with the associated distributions for incompressible flow. *E. N. Nilson* (East Hartford, Conn.).

Young, A. D., and Kirkby, S. Application of the linear perturbation theory to compressible flow about bodies of revolution. Coll. Aeronaut. Cranfield. Rep. no. 11, 8 pp. (6 plates) (1947).

Upon reconsidering the problem indicated by the title, the authors come to the same conclusions as Lees [Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1127 (1946); these Rev. 8, 108] and Sears [Quart. Appl. Math. 5, 89–91 (1947); these Rev. 8, 540]. *W. R. Sears* (Ithaca, N. Y.).

Loytsiansky, L. G. Reciprocal action of the boundary layer on the distribution of pressure over the surface of a body in a flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 205–214 (1947). (Russian. English summary)

The author indicates how it is possible to construct a plane irrotational flow of a perfect fluid (compressible in the general case) which has the same pressure distribution as the flow of a viscous fluid about a body. He obtains an expression for the resistance offered to a profile or a grid of profiles by a viscous fluid in terms of the "impulse thickness" in the wake and the projection of the vector of forces applied to the corresponding semibody or grid of semibodies in the flow of a perfect fluid. He also computes the heat produced by boundary layer friction and transferred from the solid boundary per second. *J. A. Lewis* (Providence, R. I.).

Jacob, Caius. Sur la méthode approchée de M. Lamla en dynamique des fluides compressibles. Acad. Roum. Bull. Sect. Sci. 28, 637–641 (1946).

Following E. Lamla [Luftfahrtforschung 19, 358–362 (1943); these Rev. 5, 19] the complex potential of the first approximation to the potential flow of a compressible fluid by the Rayleigh-Janzen method can be written as $f(z,\bar{z})=\varphi(x,y)+i\psi(x,y)=f_0(z)+M^2f_1(z,\bar{z})$, where $z=x+iy$, $\bar{z}=x-iy$ and M is the characteristic Mach number of the flow. The function f_1 is written as

$$f_1(z,\bar{z})=\frac{1}{4U^2}\frac{df_0}{dz}\int\overline{\left(\frac{df_0}{dz}\right)^2}\,dz+\tfrac{1}{2}\{g_1(z)-f_0(z)\},$$

where U is the characteristic velocity of the problem. Applying this method to the problem of free jets produced by two walls symmetrically situated with respect to the x-axis, the author takes U as the velocity at the free jet boundary; M is the Mach number corresponding to U. Thus f_0 is the incompressible solution. The calculation of the compressible solution then reduces to the determination of g_1. The author states that with previous results on liquid jets obtained by A. Weinstein, G. Hamel, H. Weyl, K. Friedrichs and J. Leray, g_1 usually can be uniquely determined up to a constant by the boundary conditions and by solving a Fredholm integral equation. This constant is fixed by the condition at the points of separation. When the walls are straight, the author obtains the simple result for the contraction factor of the jet as $\Gamma=\Gamma^0\{1+\frac{1}{2}M^2(1-\Gamma^0)\}$, where Γ^0 is the contraction factor for incompressible flow. *H. S. Tsien* (Cambridge, Mass.).

Eterman, I. I. Distribution of pressure over the surface of a body of revolution in a gas flow of high subsonic velocity. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 363–370 (1947). (Russian. English summary)

The author describes computational procedures which may be used in applying Christianovitch's approximate method for constructing rotationally symmetrical subsonic flows past bodies of revolution [Trudy Central. Aero-Gidrodinam. Inst., no. 481 (1940); these Rev. 7, 39]. As an example he computes the pressure-distribution on an ellipsoid of revolution with axes-ratio .215 for the stream Mach number .827. *L. Bers* (Syracuse, N. Y.).

Falkovich, S. V. Lift force of a wing of finite span. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 171–176 (1947). (Russian. English summary)

The paper deals with the pressure distribution on a plane wing of finite span in a (steady) supersonic flow. The wing is assumed to be a trapezoid whose projection on the (x, y)-plane is bounded by the straight lines $x=0$, $x=t$, $x\tan\tau\pm y=b$. The author assumes that $t/b\leqq\frac{1}{2}(M^2-1)^{-\frac{1}{2}}$, where M is the free stream Mach number. Under the usual assumptions of the linearized theory Prandtl's acceleration potential ϕ ($\phi=$constant$+\int\rho^{-1}dp$, $p=$pressure, $\rho=$density) is shown to satisfy the equation $(M^2-1)\phi_{xx}=\phi_{yy}+\phi_{zz}$ and the boundary condition $V\beta=(1/V)\int_{-\infty}^{x}\phi_z\,dx$ on the wing ($V=$free stream velocity, $\beta=$angle of attack) as well as the condition $\phi\to0$ for $x\to-\infty$. The acceleration potential is then represented in the form

$$\phi(x,y,z)=-\frac{\partial}{\partial z}(2\pi)^{-1}\int\int\frac{\gamma(\xi,\eta)\,d\xi\,d\eta}{\{(x-\xi)^2+(M^2-1)[(y-\eta)^2+z^2]\}^{\frac{1}{2}}},$$

where γ is the jump of the potential on the wing (i.e., proportional to the lift distribution) and the integration is performed along that part of the wing which lies within the Mach cone with vertex at (x, y, z). It turns out that γ is a constant if the Mach cone does not intersect a lateral edge of the wing; otherwise γ is given by a simple analytic expression. The following expressions are obtained for the lift-coefficient $C_L=\text{lift}/(\frac{1}{2}\rho_\infty V^2S)$ and the x-coordinate x_c of the center of pressure:

$$C_L=4\beta\tan\alpha\,\frac{b-\frac{1}{2}t(\tan\alpha+\tan\tau)}{b-t\tan\tau},$$

$$x_c=\frac{b-\frac{2}{3}t(\tan\alpha+\tan\tau)}{2b-t(\tan\alpha+\tan\tau)}t.$$

The formulas for a rectangular wing result by setting $\tau=0$. These formulas do not coincide with those of Schlichting [Luftfahrtforschung 13, 320–335 (1936)] which were also obtained by the method of the acceleration potential. According to the author Schlichting used a boundary condition which is correct only for the subsonic case. *L. Bers* (Syracuse, N. Y.).

Christianovich, S. A., and Yuriev, I. M. Subsonic gas flow past a wing profile. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 105–118 (1947). (Russian. English summary)

The authors present a method of constructing a circulatory flow past a closed profile under the assumption of Chaplygin's simplified equation (the so-called "$\gamma=-1$" case). They seem to be unaware of the fact that this problem was already solved by the reviewer [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 969 (1945); these Rev. 7, 497] and in a more general way by Gelbart [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1170 (1947); these Rev. 8, 417]. [Solutions identical with Gelbart's were published by Lin, Quart. Appl. Math. 4, 291–297 (1946) and by P. Germain, C. R. Acad. Sci. Paris 223, 532–534 (1946); these Rev. 8, 418, 237.] The last section of the paper discusses

Joukowsky's theorem in the framework of the Chaplygin approximation. *L. Bers* (Syracuse, N. Y.).

Gelbart, Abe. On a function-theory method for obtaining potential-flow patterns of a compressible fluid. Wartime Rep. Nat. Adv. Comm. Aeronaut., no. **L-460**, 12 pp. (1943).

This paper outlines the application of sigma-monogenic functions [Bers and Gelbart, Quart. Appl. Math. **1**, 168–188 (1943); Trans. Amer. Math. Soc. **56**, 67–93 (1944); these Rev. **5**, 25; **6**, 86] to the computation of two-dimensional subsonic adiabatic gas flows. The method was developed further in the author's subsequent paper [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1170** (1947); these Rev. **8**, 417]. *L. Bers* (Syracuse, N. Y.).

Laporte, O., and Bartels, R. C. F. An investigation of the exact solutions of the linearized equations for the flow past conical bodies. Engineering Research Institute, University of Michigan, Ann Arbor, Bumblebee Report no. **75**, 78 pp. (1948).

The authors shew that if the velocity vector is constant on all lines through the origin in a solution of the linearised equation of steady supersonic flow in the Z-direction, it must equal the real part of the vector

$$[\tfrac{1}{2}\beta\int(1+\zeta^2)F(\zeta)d\zeta,\ \tfrac{1}{2}i\beta\int(1-\zeta^2)F(\zeta)d\zeta,\ -\int\zeta F(\zeta)d\zeta],$$

where $\beta=(M^2-1)^{\frac{1}{2}}$, $\zeta=\beta(X+iY)/[Z+(Z^2-\beta^2(X^2+Y^2))^{\frac{1}{2}}]$ and $F(\zeta)$ is a regular function. Hence such problems can be solved by methods of conformal transformation. Problems at yaw and incidence are reducible to symmetrical problems by special transformations corresponding to non-Euclidean rotations within the Mach cone.

The problem of a flat symmetrical delta wing, at incidence and entirely inside the Mach cone, is solved with the boundary condition satisfied exactly, not on the zero-incidence position of the wing as is usual but on the actual wing surface, which makes the calculations considerably harder but may mean that the results (too complicated to quote) are more accurate. The condition that the lift force on the wing be finite determines the solution uniquely. This force is given as normal to the wing: in the reviewer's opinion two-dimensional subsonic analogies (relevant since the wing is inside the Mach cone) imply that the infinite suction at the leading edge will make the force more nearly perpendicular to the stream and that this will be physically correct (ignoring skin friction) when the leading edge is slightly rounded.

Lastly the flow past a symmetrically placed elliptic cone is considered and the solution expressed as a Fourier series, to obtain which an infinite system of equations must be solved. [The solution has, however, been found in a closed form, by use of Lamé function theory, by H. B. Squire, in a forthcoming paper.] *M. J. Lighthill* (Manchester).

Shen, S. F. Hypersonic flow over a slender cone. J. Math. Physics **27**, 56–66 (1948).

The hypersonic-flow approximation of Tsien [same J. **25**, 247–251 (1946); these Rev. **8**, 237] in the form suggested by Hayes [Quart. Appl. Math. **5**, 105–106 (1947); these Rev. **8**, 610] is employed. Thus the problem is identical with that of the propagation of cylindrical waves around a uniformly expanding cylinder. A transformation used by Taylor in the analogous spherical problem [Proc. Roy. Soc. London. Ser. A. **186**, 273–292 (1946); these Rev. **8**, 296] is used to reduce the nonlinear differential equation to a pair of first-order ordinary equations. The boundary conditions are the solid-boundary condition at the cone and the Rankine-Hugoniot conditions at the shock wave, which relate the flow there to stream conditions. The equations are solved numerically and results are presented in tables and graphs. They are compared with available solutions of the "exact" equations and with solutions obtained by the familiar linearized theory. *W. R. Sears* (Ithaca, N. Y.).

Stone, A. H. On supersonic flow past a slightly yawing cone. J. Math. Physics **27**, 67–81 (1948).

This paper presents in detail the theoretical basis of the calculations reported recently by Kopal and his associates [Tables of Supersonic Flow Around Yawing Cones, Massachusetts Institute of Technology, Department of Electrical Engineering, Center of Analysis. Technical Report No. 3. Cambridge, Mass., 1947; these Rev. **9**, 112]. A discussion of validity and uniqueness is included, based on the plausible assumption that the physical variables and their spatial derivatives depend continuously on the parameter ϵ [see the review of the cited tables]. Some numerical results are also given, but it is stated that the M.I.T. tables are more accurate. *W. R. Sears* (Ithaca, N. Y.).

Cherry, T. M. Flow of a compressible fluid about a cylinder. Proc. Roy. Soc. London. Ser. A. **192**, 45–79 (1947).

The author states that he has solved the problem of finding the exact solution of a two-dimensional uniform flow of a compressible perfect fluid about a cylinder. He assumes that the circulation is zero and the speed at large distances from the cylinder is subsonic, though this need not be so in the neighborhood of the cylinder. The solution contains an infinite number of parameters which theoretically can be fixed to determine the shape of the cylinder, though no indication is given how this can be accomplished. Since the author uses the solutions of Chaplygin, in the form of an infinite series of hypergeometric functions, of the linear second order partial differential equation in the hodograph variables of the potential function, this series diverges for values of the velocity whose speeds exceed the speed at infinity. The essential part of the paper is to overcome this difficulty by successfully continuing "analytically" the solutions into the region of higher speeds. This the author claims to have done, and points out that the attempt to do this by Tsien and Kuo [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **995** (1946); these Rev. **8**, 237] is based on an assumption which, in the main example of the Tsien and Kuo paper, the author shows to be false. Approximations to the exact solution in the case of the circular cylinder have been treated by Bergman [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **972** (1945); these Rev. **7**, 342].

A. Gelbart (Syracuse, N. Y.).

Miles, John W. Acoustical methods in supersonic aerodynamics. J. Acoust. Soc. Amer. **20**, 314–323 (1948).

The velocity potential due to a source distribution in the plane $z=0$ for the linearized equation of three-dimensional nonsteady supersonic flow,

$$(1-U^2c^{-2})\varphi_{xx}+\varphi_{yy}+\varphi_{zz}-2Uc^{-2}\varphi_{xt}-c^{-2}\varphi_{tt}=0$$

(U is the free-stream velocity, c the speed of sound), is obtained by means of a transformation to the wave equation $\varphi_{xx}+\varphi_{yy}+\varphi_{zz}-c^{-2}\varphi_{tt}=0$. The paper also applies the von Kármán acoustical analogy to the determination of the

wave drag of a thin symmetrical three-dimensional wing in steady supersonic flow. *E. N. Nilson.*

Panichkin, I. A. Forces acting on an oscillating profile in a supersonic gas flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 165–170 (1947). (Russian. English summary)

The author considers small harmonic oscillations of a thin slightly curved wing of infinite span in a uniform supersonic flow [cf. S. v. Borbély, Z. Angew. Math. Mech. **22**, 190–205 (1942); these Rev. **5**, 136]. Under the usual assumption of the linearized theory the velocity potential Φ satisfies the equation $N\Phi_{xx}-\Phi_{yy}+(2M/c)\Phi_{xt}+(1/c^2)\Phi_{tt}=0$, where $N^2=1-M^2$, M is the Mach number and c the speed of sound. On the wing (later to be replaced by a straight segment) Φ_y must equal the real part of $V_0(x)+V(x)e^{-ikt}$, where V_0+Ve^{ikt} is the normal component of the velocity of a point on the profile, V_0 corresponding to the vertical uniform motion and Ve^{ikt} to an oscillatory motion. Assuming steady oscillation the author sets $\Phi=\Phi_0+\phi(x,y)e^{i(\alpha t-kx)}$, where Φ_0 is the (known) solution for a nonoscillatory flow and $k=\omega M/cN^2$. For ϕ he obtains the classical equation $N\phi_{xx}-\phi_{yy}+[\omega^2/c^2N^2]\phi=0$ with the boundary condition $\phi_y=V(x)e^{-ikx}$ $(y=0)$. Integration of this equation by Riemann's method yields the formula

$$(*)\qquad \phi(x,+0)=-(1/N)\int_0^x J_0(\theta)\phi_y(t,0)dt$$

$(\theta=\sigma x-\sigma t,\ \sigma=\omega/(cN))$ for the potential along the wing. From (*) expressions for the lift and moment are obtained. These general formulas are applied to two special cases: a profile given by $y=(\beta_0+\beta_1e^{-ikt})x$ and one given by $y=\beta_0x+\beta_1e^{-ikt}$. *L. Bers* (Syracuse, N. Y.).

Kabačinskiĭ, N. N. On a question on the calculation of the profile of waves arising from the motion of a ship. Doklady Akad. Nauk SSSR (N.S.) **58**, 1301–1304 (1947). (Russian)

The problem is considered following the methods of J. H. Michell [Philos. Mag. (5) **45**, 106–123 (1898)] and L. N. Sretenskiĭ [Trudy Central. Aero-Gidrodinam. Inst., no. 319 (1937)]. The velocity potential of the disturbed flow is obtained as a velocity potential corresponding to an infinite acceleration of gravity plus a corrective term which takes into account its finite magnitude. Both terms are expressed as definite integrals. A numerical example involving the use of the Neumann function of the first kind is given. This example contains numerical tables of

$$M(x,z)=\int_1^\infty (u^2-1)^{-\frac{1}{2}}\exp(-zu)\cos(xu)du$$

for $x=0(1)19$ and $z=0,\ 0.25,\ 0.5,\ 1,\ 2$. A very short table of two double integrals with integrands of the same type as in M is also included. *I. Opatowski* (Chicago, Ill.).

Holtsmark, J. The absorption and diffusion of sound by resonators. Avh. Norske Vid. Akad. Oslo. I. **1946**, no. 6, 52 pp. (1947).

The author studies the effect of an acoustic field on resonators and oscillators. He assumes that the incident wave consists of a large beam of parallel rays. When the oscillator is kept immovable, the disturbance of the original beam by reflection and diffraction is termed a "reflected wave." Actually, the incident beam causes the oscillator to vibrate so as to set up "re-radiated" waves in the surrounding medium. Furthermore, part of the incident ray is absorbed by the frictional forces of the oscillator. The energy exchange is generally expressed in terms of mechanical and radiative impedances, as well as in terms of absorption and diffusion coefficients. In treating the total absorption of an oscillator in a beam of given frequency distribution, the author tacitly assumes the various monochromatic components to be incoherent. The general theory is applied to several special cases, including the "pocket resonator," the Helmholtz resonator in a plane wall, the Rayleigh disk and an elastic circular membrane in an infinite plane screen. An extension of the Rayleigh disk problem is given so as to include the case where the disk radiates opposite an infinite plane wall parallel to that of the disk itself. Other topics treated concern rectangular membranes in the end wall of an infinite guide of rectangular cross-section and rectangular and circular membranes in the wall of a rectangular box. Use is made of Bessel-function expansions.

[In a communication to the reviewer, the author indicates several misprints in his basic formulae. The following are noteworthy, particularly so since the author's definitions are at variance with corresponding American definitions; in equations (2.6) and (2.7) a 4 should be added in the numerator, in equations (2.8) and (2.9) a factor S^2; while the 2 in the denominator of equation (2.9) should be deleted. Furthermore, the right-hand sides of equations (2.12) and (3.3) should have their signs reversed.]

C. J. Bouwkamp (Eindhoven).

Elasticity, Plasticity

Krylov, V. V. On the application of complex variables to plane problems of the theory of elasticity with finite deformations. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 81–86 (1948). (Russian)

In this article, which is a sequel to a previous one by the author [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 647–656 (1946); these Rev. **8**, 357] complex variables and functions are introduced. Some applications are given. *H. A. Lauwerier* (Amsterdam).

Narodeckiĭ, M. Z. The determination of the stresses in a circular ring under the influence of a concentrated force. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 7–18 (1948). (Russian)

An elastic medium fills a circular ring in the complex z-plane. The boundaries of the ring are given by $|z|=R_2$ and $|z|=R_1$ with $R_2>R_1$. The determination of the state of elastic equilibrium when two mutually equilibrating forces are applied at the points $z=\pm R_2$ of the ring, by the Fourier expansion method, leads to slowly converging series when R_1/R_2 is nearly 1. The author provides an effective solution of the problem by employing a method of successive approximation based on G. V. Kolossoff's and N. I. Muschelišvili's method of solving two-dimensional elastic problems with the aid of complex variable theory. *I. S. Sokolnikoff.*

Richter, H. Bemerkung zum Moufangschen Verzerrungsdeviator. Z. Angew. Math. Mech. **28**, 126–127 (1948).

For infinitesimal strains the decomposition of the strain tensor into the strain deviation (representing the change of shape) and a spherical tensor (representing the change of volume) is well known. For finite strains a formally similar

decomposition has been suggested by R. Moufang [same Z. 25/27, 209–214 (1947); these Rev. 9, 315]. The author shows that Moufang's strain deviation cannot be considered as an adequate representation of the change of shape: a similarity transformation following a transformation which involves a change of shape does not leave Moufang's deviation invariant. *W. Prager* (Providence, R. I.).

Jurmain, Jacob Harry. Applications of tensor analysis to elasticity and piezoelectricity. J. Franklin Inst. **245**, 475–500 (1948).

Rabotnov, Yu. N. The equilibrium of an elastic medium with after-effect. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 53–62 (1948). (Russian)

The author points out that Volterra's integral representation of elastic after-effects can be formally obtained from Hooke's law by replacing the elastic constants by appropriate integral operators. This makes it possible to treat problems concerning materials with after-effect by introducing these operators into the solutions of the equivalent elastic problems. Several examples are discussed. *W. Prager* (Providence, R. I.).

Gross, B. On creep and relaxation. J. Appl. Phys. **18**, 212–221 (1947).

The response to stress of a linear viscoelastic material is mathematically analogous to the transient response of a capacitive-resistive electrical network or transmission line. This behavior can be analyzed in terms of a mechanical model (i.e., a set of first-order differential equations), a single differential operator equation of higher order, a distribution of elastic relaxation times or a distribution of elastic retardation times, or in other ways. This paper analyzes the mathematical connections among various methods of analyzing the viscoelastic response. In particular, Laplace transforms are utilized to develop relations between creep and relaxation, distributions of elastic relaxation and retardation times are derived from creep and relaxation curves, and relations are developed between the two types of distribution function. Particular creep and relaxation functions are discussed. *T. Alfrey, Jr.*

Riz, P. M. Large deformations and plasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 211–212 (1948). (Russian)

The linear theory of elasticity becomes unsatisfactory for large strains or for materials obeying nonlinear stress-strain laws. The author states that previous investigators restricted themselves to problems where only one of these two causes of nonlinearity needs to be considered at a time. The present paper is concerned with more general problems where this simplification is not admissible. The discussion is based on a stress-strain law which contains quadratic terms in the finite Eulerian strain components. [While the author repeatedly refers to problems of plasticity, his stress-strain law essentially represents nonlinear elastic behavior.] *W. Prager* (Providence, R. I.).

Berezancev, V. G. The limit equilibrium of a substance with internal friction and cohesion under axially symmetric stress. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 95–100 (1948). (Russian)

The author discusses axially symmetric distributions of stress in a plastic material such that the Mohr circles representing states of stress at the yield limit have two straight lines as envelope. The problem is made statically determinate by the convenient but rather arbitrary assumption that the hoop stress equals one of the principal stresses in the meridional plane. The characteristic equations are used as the basis of a method of numerical integration. *W. Prager* (Providence, R. I.).

Prager, William. Plasticity for the aerodynamicist. J. Aeronaut. Sci. **15**, 253–262 (1948).

Bahšiyan, F. A. On the visco-plastic flow under the impact of a cylinder on a plate. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 47–52 (1948). (Russian)

The paper is concerned with the following problem. A circular plate made of a visco-plastic material and built in at its edge $r=b$ is at rest until the central portion $r<a$ is suddenly given the transverse velocity V; the ensuing transverse displacement w of the plate is sought as function of the radius r and the time t. The solution presented in the paper is based on the assumption that all stresses but the shearing stresses τ_{rz} are negligible. *W. Prager.*

Sokolovskiĭ, V. V. Plastic stresses in rotating disks. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 87–94 (1948). (Russian)

The paper is concerned with the stresses in a rotating disk with step-profile in the case where the angular velocity is sufficiently large to cause permanent deformations. The disk material is supposed to be strain-hardening and to follow stress-strain relations of the Hencky-Nadai type (deformation type). [As A. A. Ilyushin has shown [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 347–356 (1946); these Rev. **8**, 240], the use of stress-strain relations of this type is justified only if the directions of the principal axes and the ratios between the principal components of the stress deviator remain constant during the loading process. While the first of these conditions is fulfilled for the present problem, the second is not. The validity of the author's results may therefore be questioned.] *W. Prager* (Providence, R. I.).

Sokolov, A. P. On an elastic-plastic state of a plate. Doklady Akad. Nauk SSSR (N.S.) **60**, 33–36 (1948). (Russian)

The paper is concerned with a state of plane elastic-plastic stress in a thin infinite plate with a circular hole. The given state of stress at infinity is assumed to differ but little from a state of plane isotropic tension. The "solution" given in the paper corresponds to the first perturbation of the rotationally symmetric case which would result from plane isotropic tension at infinity. *W. Prager.*

Ling, Chih-Bing. The stresses in a plate containing an overlapped circular hole. J. Appl. Phys. **19**, 405–411 (1948).

The plane-stress distribution in a plate containing a lenticular hole is investigated. The stress functions making up the solution are found in a bipolar coordinate system and the boundary conditions are resolved with the aid of Fourier transforms. Numerical results are obtained and the stress concentration is plotted for various degrees of "overlap" of the circles which form the hole. *G. F. Carrier.*

Green, A. E. Three-dimensional stress systems in isotropic plates. I. Philos. Trans. Roy. Soc. London. Ser. A. **240**, 561–597 (1948).

The paper is devoted to a general analysis of the three-dimensional stress distributions in a flat, elastic, infinite plate containing a circular hole. The plate is supposed to be of finite thickness and its faces are free from applied stress. The problem of a plate subjected to a uniform tension parallel to the faces of the plate, when the hole is free from applied stress, is considered in detail. In calculations the diameter of the hole is assumed equal to the thickness of the plate. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Southwell, R. V. Relaxation methods applied to engineering problems. XIII. The flexure and extension of perforated elastic plates. Proc. Roy. Soc. London. Ser. A. **193**, 147–171 (1948).

Reissner, Eric. Note on the membrane theory of shells of revolution. J. Math. Physics **26**, 290–293 (1948).

The author establishes the connection between the stress functions proposed by A. Pucher [Beton und Eisen **33**, 298–304 (1934); Proc. 5th Int. Congr. Appl. Mech., 1939, pp. 134–139] and P. Nemenyi [Bygningsstatiske Meddelelser **8**, 53–72 (1936)] and P. Nemenyi and C. Truesdell [Proc. Nat. Acad. Sci. U. S. A. **29**, 159–162 (1943); these Rev. **5**, 84] for the treatment of problems of the membrane theory of shells. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Föppl, Ludwig. Schiefe Kreiskegelschale unter Innen- oder Aussendruck. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 1–17 (1944).

The author determines the membrane solution for the stresses in a thin shell with middle surface $xz=c(x^2+y^2)$, when the applied load is a uniform normal pressure on the wall. *E. Reissner* (Cambridge, Mass.).

Muštari, H. M. Invariant equations of equilibrium of the boundary zone of an elastic shell in complex form. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 129–136 (1948). (Russian)

The author follows the customary procedure [see, for example, W. Z. Chien, Quart. Appl. Math. **1**, 297–327 (1944); these Rev. **5**, 195] to formulate a system of six invariant macroscopic equations of shell theory. An assumption of the Kirchhoff-Love hypothesis and a definition of the complex stress tensor in terms of the usual macroscopic stress tensor and the quantities depending on the geometry of the middle surface enable the author to write a set of approximate equilibrium equations on the complex stress tensor. The goodness of approximation in these equations depends on the thickness of the shell. It is shown that the complex stress tensor may be obtained from a stress function satisfying a certain fourth-order partial differential equation. The simplifications resulting from an assumption that the surface of the shell is developable are indicated in some detail. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Gutman, S. G. A general solution of the plane problem of the equilibrium of a beam bounded by a logarithmic spiral. Doklady Akad. Nauk SSSR (N.S.) **60**, 559–562 (1948). (Russian)

By introducing a system of curvilinear coordinates connected with the Cartesian coordinates by the formula $\alpha+i\beta=e^{\delta}\log z$, $z=x+iy$, δ a parameter, the author obtains a solution of the biharmonic equation (Airy's stress function) which can be used to solve several problems of the equilibrium of curved beams bounded by logarithmic spirals. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Chitty, Letitia. On the cantilever composed of a number of parallel beams interconnected by cross bars. Philos. Mag. (7) **38**, 685–699 (1947).

Conway, H. D. Calculation of frequencies of truncated pyramids. Development of the theories for the longitudinal, flexural and torsional vibrations of a truncated cone. Aircraft Engrg. **20**, 148 (1948).

Beránek, Jiří. Sur le spectre de vibration de la membrane de Routh. C. R. Acad. Sci. Paris **226**, 1244–1246 (1948).

By considering string networks approximating a rectangular membrane, and transforming by means of eigenfunctions, the author computes frequencies, frequency densities and energy densities. *P. Franklin.*

Scholte, J. G. On Rayleigh waves in visco-elastic media. Physica **13**, 245–250 (1947).

The author criticizes the paper on this subject which was published by E. Hardtwig [Z. Geophys. **18**, 1–20 (1943); these Rev. **8**, 548], first on the score of unnecessary restriction of the definition of Rayleigh waves in a visco-elastic medium and, secondly, of a mistaken interpretation of the roots he derives from his Rayleigh equation.

Incidentally the author presents an excellent summary of visco-elastic wave theory in the terse and transparent language of modern vectors. *J. B. Macelwane.*

Hardtwig, Erwin. Über die Anfangswertaufgabe in der Theorie der Rayleighwellen. Z. Angew. Math. Mech. **25/27**, 1–13 (1947). (German. Russian summary)

The author develops further the theory of Rayleigh waves he had set up in his earlier papers [Z. Geophys. **18**, 1–20, 164–180 (1943); these Rev. **8**, 548], and obtains a more general integral by superposition of the particular integrals, first in two dimensions and then in three, on the assumption of prescribed initial conditions. The work seems to be open to the same objections as were raised by Scholte to Hardtwig's previous papers [cf. the preceding review]. *J. B. Macelwane* (St. Louis, Mo.).

Sveklo, V. A. Plane waves and Rayleigh waves in an anisotropic medium. Doklady Akad. Nauk SSSR (N.S.) **59**, 871–874 (1948). (Russian)

The paper deals with the propagation of plane elastic waves in an anisotropic half-plane characterized by four elastic constants. From the class of solutions of the wave equation, the arbitrary function of which is also a solution of the wave equation, the author deduces an analogue of Rayleigh's surface waves for a two-dimensional anisotropic medium. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Sveklo, V. A. The sources of vibration in an anisotropic half plane. Doklady Akad. Nauk SSSR (N.S.) **59**, 1077–1080 (1948). (Russian)

A continuation of the paper reviewed above, dealing with the study of the singularity of the solution corresponding to a point source of disturbance in an anisotropic elastic half-plane. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Post, E. J. Reciprocal properties of elastic waves in anisotropic media. Nederl. Akad. Wetensch., Proc. **51**, 65–72 = Indagationes Math. **10**, 36–43 (1948).

The author notes that the work of B. van Dijl [Physica **3**, 317–326 (1936)] implies the existence of reciprocal directions in quartz. Reciprocal directions are such that one direction corresponds to displacement and the other to direction of propagation of an elastic wave, and these properties are interchangeable for the two directions. The aim of the present paper is to give an analysis of reciprocal directions for a general homogeneous anisotropic elastic body, for which stress (T^{ij}) and strain (s_{kl}) are related by $T^{ij}=c^{ijkl}s_{kl}$, the coefficients being elastic constants. The equations of small vibrations are simplified by assuming that amplitude, its rate of change, wave length and frequency all undergo only small changes for displacements in time and space of one period and one wave length, respectively. Through investigation of energy flow the author is led to define the energy function of two vectors n^k, p^k: $E=c^{ijkl}p_ip_jn_kn_l$. If the magnitudes of the two vectors are held fixed, stationary values of E correspond to vector pairs satisfying $Ep^i=c^{ijkl}n_jp_kn_l$, $En^j=c^{ijkl}p_ip_kn_l$. Each such vector pair is reciprocal. The vector of the flow of energy is perpendicular to the wave front and the velocities of reciprocal waves are equal and have an extreme value. The existence of self-reciprocal directions imposes conditions on the elastic constants. Self-reciprocal and reciprocal directions are tabulated for various crystal types. The paper ends with a discussion of the piezo-electric resonator. *J. L. Synge.*

Lifšic, I. M. The scattering of short elastic waves in a crystal lattice. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 293–300 (1948). (Russian)

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Herzberger, M. Performance of an optical system as the sum of the contributions of its construction elements. J. Opt. Soc. Amer. **38**, 324–328 (1948).

In a former paper of the author [same J. **26**, 197–204 (1936)] a new theory of image formation was developed, based upon the concept of diapoints. The diapoints which belong to a given object point are those points of the meridional plane through the object point where the rays from the object point intersect this plane. The structure of this point set yields a simple and significant characterization of the image formation of the instrument. In the present paper a detailed description of design methods is given in which the concept of diapoints is used. Formulae are given to show how the diapoint deviations can be considered as the sum of deviations originating at the single surfaces. With the aid of graphical illustrations of these formulae the designer may obtain accurate information about the effect of the single surfaces on the rays traversing the system.

R. K. Luneburg (New York, N. Y.).

Marx, H. Über eine neue Darstellung der Bildfehler 3. Ordnung. I. Theoretischer Teil. Optik **2**, 364–381 (1947).

The author introduces new variables into the Seidel sums which describe the errors of third order. The new variables are invariant with respect to the transition from one surface to another and if determined permit the computation of the construction data. The formulas derived by Seidel have a simpler appearance. They contain, however, parameters which are not independent. The author succeeds in deriving a system of independent parameters. *M. Herzberger.*

Mertè, W. Plankonvexe und plankonkave, ideal anastigmatische Einzellinsen. Optik **1**, 389–394 (1946).

A lens ideally corrected for astigmatism is, according to the author, a lens for which astigmatism is corrected on all the rays through the center of the diaphragm opening for the infinite plane. The author tries to find a lens such that this condition is fulfilled under the condition that the first surface of the lens is plane and that the diaphragm is in image space. The second surface obtained proves to be a surface of second order determined by the focal length of the lens and the position of the exit pupil. [In the paper there is a discrepancy between drawing and text. The angle designated by γ in the drawing is designated in the text by v.]

M. Herzberger (Rochester, N. Y.).

Buchdahl, H. A. Algebraic theory of the primary aberrations of the symmetrical optical system. J. Opt. Soc. Amer. **38**, 14–19 (1948).

The author considers the well-known Seidel theory of optical aberrations. His principal contribution consists in an appropriate choice of variables. He considers a ray as given by two vectors, the vector **a** from the vertex to the intersection point of the ray with the plane tangential to the vertex, and a vector **S** whose components are the ratios ξ/ζ, η/ζ, where ξ, η, ζ are the direction cosines of the ray. The formulas which the author obtains are equivalent to Seidel's theory, but the general invariants developed in the first part of his article are of value for the general theory of aberrations. *M. Herzberger* (Rochester, N. Y.).

Lévy, Maurice. Sur la propagation de la lumière dans un milieu dispersif. C. R. Acad. Sci. Paris **226**, 236–238 (1948).

In the molecular theory of the propagation of light in an isotropic nonabsorbent dispersive medium the effective field **E'**, i.e., the field acting on each molecule when account is taken of the effect of all the others, satisfies the singular integral equation

$$\mathbf{E}'(r,t)=\mathbf{E}_0(r,t)+\text{rot rot}\int_\Omega^{R-1}N\alpha\mathbf{E}'(r',t-R/c)d\Omega(r'),$$

where $\mathbf{E}_0$ is the incident field, α the polarisability of a molecule, N the number of molecules per c.c., $R=|r-r'|$. The author summarises the analytical defects in previous attempts of Oseen [Ann. Physik (4) **48**, 1–56 (1915), p. 15], Lundblad [Ann. Physik (4) **57**, 183–202 (1918)], Ewald [Ann. Physik (4) **49**, 1–38, 117–143 (1916)] to deal with particular cases of this equation and outlines a method by which he claims that a rigorous solution can be obtained. The indications given are too brief to allow the reviewer to form an opinion on the validity of this claim.

E. H. Linfoot (Bristol).

Holl, H. Lichtstreuung an dielektrischen Kugeln vom Brechungsexponenten $n=4/3$. Optik **1**, 213–226 (1946).

This paper deals with the scattering of light by dielectric spheres suspended in air (water drops in clouds). The inten-

sity and the polarization of the scattered light depend on the radius ρ, the index of refraction n of the spheres, the wave length of light and the angle φ with the incident beam. Theoretical formulae for the scattering function have been derived by G. Mie [Ann. Physik (4) 25, 377–445 (1908)]. The author evaluates these formulae numerically, assuming $n=4/3$, for values of $\alpha=2\pi\rho/\lambda$ ranging from 0.3 to 4.2. The intensity and polarization of the scattered light are tabulated as functions of the angle φ. *R. K. Luneburg.*

Kašpar, Emil. Systèmes de réflexions et de réfractions. Rev. Optique 27, 15–28 (1948).

A system of (refracting and reflecting) plane surfaces is called a prism. The author gives a concise and very useful formula for computing the direction of an emerging ray after a number of reflections and refractions. Let $\mathfrak{n}_1, \cdots, \mathfrak{n}_k$ be the unit vectors normal to the first, $\cdots$, kth surfaces. Let $\mathfrak{s}$, $\mathfrak{s}'$ be the unit vectors in object and image space; μ, μ' the refractive indices; analogously $\mathfrak{s}_{\nu,\nu+1}$ and $\mu_{\nu,\nu+1}$ unit vector and refractive index in the space between the νth and $(\nu+1)$th surfaces. We find $\mathfrak{s}$ with the help of the determinant

$$\mathfrak{s}'=\frac{\mu}{\mu'}\begin{vmatrix} \mathfrak{s}, & (\mathfrak{s}\cdot\mathfrak{n}_1)-\frac{\mu_2}{\mu_1}(\mathfrak{s}_{12}\cdot\mathfrak{n}_1), & \cdots, & (\mathfrak{s}\cdot\mathfrak{n}_k)-\frac{\mu'}{\mu}(\mathfrak{s}'\cdot\mathfrak{n}_k) \\ \mathfrak{n}_1, & 1, & (\mathfrak{n}_1\cdot\mathfrak{n}_2), \cdots, & (\mathfrak{n}_1\cdot\mathfrak{n}_k) \\ \mathfrak{n}_2, & 0, & 1, \cdots, & (\mathfrak{n}_2\cdot\mathfrak{n}_k) \\ \cdots & \cdots & \cdots & \cdots \\ \mathfrak{n}_k, & & \cdots, & 1 \end{vmatrix},$$

where

$$(\mathfrak{s}_{\nu,\nu+1}\cdot\mathfrak{n}_k)=\{1-(\mu_{\nu,\nu-1}/\mu_\nu)^2+(\mu_{\nu,\nu+1}/\mu_\nu)^2(\mathfrak{s}_{\nu,\nu-1}\cdot\mathfrak{n}_k)^2\}^{\frac{1}{2}}.$$

The formula simplifies considerably for reflections. A series of examples complete the paper. *M. Herzberger.*

Abelès, Florin. Sur la suppression du facteur de réflexion des substances absorbantes par recouvrement d'une ou deux couches minces transparentes. C. R. Acad. Sci. Paris 226, 1259–1261 (1948).

Tuckerman, L. B. Multiple reflections by plane mirrors. Quart. Appl. Math. 5, 133–148 (1947).

Quaternion treatment is applied to the problem of multiple plane reflections. In addition to familiar basic theorems, two on nondeviating systems are demonstrated. Suggestions are given for computation of the resultant of several reflections. Illustrative applications are made to a five-mirror range-finder combination and to a Leman-Springer nondeviating erecting prism. *A. J. Kavanagh.*

Coxeter, H. S. M. The product of three reflections. Quart. Appl. Math. 5, 217–222 (1947).

Characteristics of the product of three plane reflections are discussed in terms of the geometry of one of the triangles cut out by the planes on a unit sphere centered at their common point, and in terms of the pedal triangle of the former triangle. The case of three mirrors forming a prism is also considered. The relation is shown to results by Synge [Quart. Appl. Math. 4, 166–176 (1946); these Rev. 7, 532] and by Tuckerman [cf. the preceding review]. *A. J. Kavanagh* (Buffalo, N. Y.).

Wunderlich, Walter. Spiegelung am elliptischen Paraboloid. Monatsh. Math. 52, 13–37 (1948).

The author investigates the reflection on an elliptic paraboloid as seen from a given point Z. This point is assumed to be the point at infinity of the paraboloid. The lines which are reflected into Z form a congruence. Three lines of this congruence pass through each point in space and four lie in each plane. After three consecutive reflections a ray parallel to the axis of the paraboloid, but outside the principal planes, becomes parallel to the axis again. Therefore any point has three primary, three secondary and one tertiary reflections on the paraboloid. The reflections of points and lines are studied in detail. The paper is partly an elaboration of results obtained by H. Horninger [Monatsh. Math. Phys. 45, 53–75 (1936)]. *E. Lukacs.*

Melkich, Alexander. Ausgezeichnete astigmatische Systeme der Elektronenoptik. I. Akad. Wiss. Wien, S.-B. IIa. 155, 393–438 (1947).

The author discusses the image formation of electron optical instruments if the index of refraction has two planes of symmetry at right angles to each other. Simple cylindrical lenses, crossed cylindrical lenses and also rotational symmetrical fields are the most important special examples. First it is shown that electromagnetic fields of this type can be characterized by five analytic functions $\phi(z)$, $D(z)$, $G(z)$, $K(z)$, $L(z)$, provided that the discussion of aberrations is restricted to third order terms. Then the first order or Gaussian optics of such fields is discussed along the usual lines. The associated differential equations of the paraxial rays are linear and depend only on the functions ϕ, K, D. Conditions are given for stigmatic images of points on the axis. A corresponding inhomogeneous system of differential equations is satisfied by the third order aberrations, where the inhomogeneous terms depend only on the associated paraxial rays. These equations are solved and integral expressions for the 20 coefficients of the aberration polynomials are given. *R. K. Luneburg* (New York, N. Y.).

Melkich, Alexander. Ausgezeichnete astigmatische Systeme der Elektronenoptik. II. Akad. Wiss. Wien, S.-B. IIa. 155, 439–471 (1947).

[Cf. the preceding review.] The assumption of two planes of symmetry, at right angles to each other, is made for the electromagnetic field itself instead of for the index of refraction as in the first part of the paper. It is shown that the former assumption cannot be considered as a special case. However, it is possible to obtain cylindrical lenses, crossed cylindrical lenses and lenses of rotational symmetry by suitable specialization also from this group of electromagnetic fields. The author first shows that fields of this general type can be characterized by six scalar functions instead of by five as in the former case. He then discusses the first order optics in such systems and derives integral expressions for 40 coefficients of the third order aberration polynomials. A geometrical discussion of the different types of aberrations is given. *R. K. Luneburg* (New York, N. Y.).

Koch, Jørgen. A special case of velocity focusing in a magnetic deflecting field. Danske Vid. Selsk. Mat.-Fys. Medd. 24, no. 7, 22 pp. (1948).

If, for purposes of mass-spectrographic analysis, a narrow beam of ions is deflected by a magnetic prism, the variations ΔV of the accelerating potential V will cause disturbing fluctuations of the beam at the collecting cylinder. The author shows that this movement may be compensated for by making the beam before entering the magnetic field pass through a deflecting condenser. One plate of this condenser is grounded, the other is supplied with a voltage propor-

tional to ΔV. The conditions for compensations are derived and applied to the design of mass spectrographs.

R. K. Luneburg (New York, N. Y.).

Kofink, W. Reflexion elektromagnetischer Wellen an einer inhomogenen Schicht. Ann. Physik (6) **1**, 119–132 (1947).

The author is concerned with the study of electromagnetic waves travelling through a homogeneous medium of which the dielectric constant is a real-valued function of the coordinate z. While discussing previous work of P. S. Epstein [Proc. Nat. Acad. Sci. U. S. A. **16**, 627–637 (1930)] and K. Rawer [Ann. Physik (5) **35**, 385–416 (1939)] he constructs functions $\epsilon(z)$ for which the corresponding medium is nonreflective. Guided by the theory of P. H. van Cittert, the author derives a general Ricatti equation for the ratio of the amplitudes of reflected and incoming waves, for the usual two cases of polarization. A connection is established between this Ricatti equation and the differential equation of W. Geffcken [Ann. Physik (5) **40**, 385–392 (1941); these Rev. **6**, 166] for the reflection function. The remainder of the paper is devoted to discussions in connection with the theories of R. Gans [Ann. Physik (4) **47**, 709–736 (1914)] and S. Brodetsky [Proc. Edinburgh Math. Soc. (1) **34**, 45–60 (1916)].

C. J. Bouwkamp (Eindhoven).

Maréchal, André. Étude des effets combinés de la diffraction et des aberrations géométriques sur l'image d'un point lumineux. Rev. Optique **26**, 257–277 (1947).

The paper reports the first of three parts of an investigation on this subject, and deals with the theory. Given the wave front emerging from an optical system and a point C in the image space, a mean sphere of center C is defined such that the mean-square departure E_0 of the wave from the sphere is a minimum with respect to the radius of the sphere. Assuming uniform amplitude distribution over the wavefront, a lower limit for the illumination e at C is given by $e>[1-(2\pi^2/\lambda^2)E_0]^2$. This relation is used to study variations of intensity in the image in the neighborhood of the point of maximum intensity, on the assumption that E_0 is small, so that the above inequality can be treated substantially as an equality. Expressions are derived for tolerances for spherical, coma, astigmatism and curvature of field, on the assumption of a standard value of E_0 corresponding roughly to the Rayleigh limit.

A. J. Kavanagh.

Maréchal, André. Étude des effets combinés de la diffraction et des aberrations géométriques sur l'image d'un point lumineux. II. Rev. Optique **27**, 73–92 (1948).

[Cf. the preceding review.] This part describes the principle and construction of a mechanical integrator for calculating diffraction integrals in the presence of geometrical aberrations. It is assumed that the wave emerging from the optical instrument can be described by the equation

$$\Delta=f_0(\rho^2)+(\rho^2/2R^2)x'+(\rho\rho_1/R)\cos(\varphi-\varphi_1)+\rho f_1(\rho^2)\cos\varphi+f_2(\rho^2)\cos 2\varphi,$$

where Δ is the departure of the wave from a sphere centered at the point at which illumination is to be calculated; R is the radius of the sphere; ρ and φ are polar coordinates in the exit pupil; x', ρ_1 and φ_1 are cylindrical coordinates of the center of the sphere with respect to the paraxial focus; $f_0(\rho^2)$ represents the spherical aberration, $f_1(\rho^2)$ the coma and $f_2(\rho^2)$ the astigmatism. Integration is carried out along an Archimedes spiral in the exit pupil. Examples are given of the use of the instrument.

A. J. Kavanagh.

Mirimanov, R. G. A solution by means of Laguerre functions of the problem of the diffraction of a plane electromagnetic wave by an infinite parabolic reflector. Doklady Akad. Nauk SSSR (N.S.) **60**, 203–206 (1948). (Russian)

The general solution of the wave equation is written in terms of products of Laguerre functions of parabolic coordinates. The general solution of Maxwell's equations is written by introducing two scalar potential functions. The formal solution of the diffraction problem is obtained by finding expressions for the coefficients in the infinite series.

W. H. Furry (Cambridge, Mass.).

Mirimanov, R. G. A solution of the problem of the diffraction of a spherical electromagnetic wave by an infinite parabolic reflector. Doklady Akad. Nauk SSSR (N.S.) **60**, 357–360 (1948). (Russian)

The method described in the paper reviewed above, employing Laguerre functions of parabolic coordinates, is used to obtain the formal solution for the case of an infinitesimal electric dipole located at the focus of the paraboloid and oriented along the axis.

W. H. Furry.

Risco, M. Le principe de Huyghens dans l'optique des corps en mouvement. J. Phys. Radium (8) **8**, 282–288 (1947).

If an observer A finds that the wave length of plane waves of light is λ_0, an observer B, in uniform motion relative to A, will find that the wave length is

$$(*)\qquad \lambda=\lambda_0(c^2-v^2)^{-1}(c+v\cos\theta),$$

where v is the relative velocity and θ is the angle between the direction of relative motion and the direction of propagation of the waves as observed by B. Since Maxwell's equations are invariant under the Lorentz transformation, one would expect that, in dealing with the optics of moving bodies, one would use the ordinary Huyghens spherical wavelet construction in all frames of reference in uniform relative motion. The present author claims that it is better not to do so. His argument seems to be that in the Huyghens construction, observer A above uses spheres $r=n\lambda_0$. As observer B must take into account variation of wave-length with direction of relative motion as given in (*), he must use the limaçon

$$(**)\qquad r=n\lambda_0(c^2-v^2)^{-1}(c+v\cos\theta)$$

(which is not a wave-surface) as the Huyghens wavelet. The justification of this assertion is not obvious to the reviewer.

E. T. Copson (Dundee).

Riblet, H. J., and Barker, C. B. A general divergence formula. J. Appl. Phys. **19**, 63–70 (1948).

The authors determine the amount of energy reflected in any direction from a smooth nonabsorbent object placed in an electromagnetic wave. Let d_1 be the distance from the source P to the point of reflection R, and d_2 that from R to the point of observation Q. Further, let ρ_1 and ρ_2 denote the principal radii of curvature of the object's surface at R, Ω the angle of incidence, θ_1 and θ_2 the angles between the incident ray and the two principal directions on the surface at R. If D denotes the ratio of the reflected-energy density per unit area (taken normal to RQ) at Q, to the energy density per steradian (with respect to P) incident on the object at R, the authors obtain

$$D=\frac{\rho_1\rho_2\cos\Omega}{\{4d_1^2d_2^2+(d_1+d_2)^2\rho_1\rho_2\}\cos\Omega+2d_1d_2(d_1+d_2)(\rho_1\sin^2\theta_1+\rho_2\sin^2\theta_2)}.$$

The derivation of this formula is elementary and purely geometrical. From the physical point of view the expression will be an excellent approximation so long as the wavelength is small compared to ρ_1 and ρ_2. The formula includes many special cases previously dealt with by various authors.
C. J. Bouwkamp (Eindhoven).

Durand, Émile. Champ électromagnétique d'une charge en mouvement. Cahiers de Physique nos. 31–32, 75–82 (1948).

This paper is an elaboration of R. Reulos's idea [same Cahiers, no. 3, 1–14 (1941); these Rev. 8, 181] that a moving charge can be thought of as a series of decaying stationary charges. It is shown that the totality of the elementary fields produced by these decaying charges is equivalent to the field calculated by the usual procedure of retarded potentials. *C. Kikuchi* (East Lansing, Mich.).

Aymerich, Giuseppe. Sul moto piano di un corpuscolo ellettrizzato in un particolare campo magnetico di tipo notevole. Rend. Sem. Fac. Sci. Univ. Cagliari 16 (1946), 15–28 (1948).

Wendt, Georg. Zur Theorie der Ablenkfehler von Elektronenstrahlbündeln, insbesondere in gekreuzten Ablenkfeldern. Ann. Physik (6) 1, 83–94 (1947).

The author discusses the path of an electron stream in a system of crossed electromagnetic fields. It is assumed that one field is in the (x, z)-plane, the other in the (y, z)-plane, with the z-axis as an axis of symmetry. The method used is to expand the total scalar and vector potentials as double power series in x and y, and then apply Fermat's principle to obtain the electron path. Most of the paper is devoted to a discussion of the order of magnitude of the various terms in the series thus obtained, with explicit expressions for the leading terms in the deviation of the path of the electrons. *M. C. Gray* (New York, N. Y.).

Flamm, L. Elektronen-Feldtheorie. Österreich. Ing.-Arch. 1, 358–371 (1947).

A theory in which the electrons are regarded as field matter filling all space and the magnetic field as the velocity field of the electric field matter is given. The theory is developed along the lines suggested by H. Bateman [Philos. Mag. (6) 34, 405–423 (1917)]. The latter considered the electromagnetic field from the standpoint of moving electric and magnetic lines of force. *C. Kikuchi.*

Flamm, Ludwig. Elektrische Feldmechanik. Acta Physica Austriaca 1, 259–284 (1948).

The author introduces the idea of an "elementary electric field," which is the primary field associated with the elementary electric charge. This has inertial and elastic properties similar to those of a rigid body and thus represents a form of matter which is called "field matter." For a homogeneous elementary electric field he considers transverse vibrations of the electric field line tubes and concludes that the magnetic field has no material existence, but that the magnetic field vector can be associated with surface paths and velocities of the electric lines.

The author next discusses the differential equation for the motion of the electric field line tubes, and shows that a transverse wave is propagated along the tube with the velocity of light. This is called the "elementary electric wave." For the homogeneous elementary electric field the elementary electric wave is a plane wave, which gives rise to an electromagnetic field satisfying Maxwell's equations in source-free regions. Formulas for the kinetic energy and the impulse density are obtained, and interpreted to require use of the dynamics of mass lines rather than of mass points in studying the properties of the field matter.
M. C. Gray (New York, N. Y.).

Podolsky, Boris, and Schwed, Philip. Review of a generalized electrodynamics. Rev. Modern Physics 20, 40–50 (1948).

The authors give a summary of a linear fourth-order electrodynamics developed by B. Podolsky [Physical Rev. (2) 62, 68–71 (1942); these Rev. 4, 31] and others. [For references, see A. E. S. Green, Physical Rev. (2) 72, 628–631 (1947); these Rev. 9, 69.] The auxiliary condition is discussed in greater detail than has been done previously.
C. Kikuchi (East Lansing, Mich.).

Stevenson, A. F. Theory of slots in rectangular waveguides. J. Appl. Phys. 19, 24–38 (1948).

The author develops a theory of the effect on wave propagation due to a narrow rectangular slot cut in one of the sides of an infinite rectangular wave guide with perfectly conducting wall of negligible thickness. It is assumed that the guide transmits the dominant wave only, while the length of the slot is approximately equal to half the wavelength. As a preliminary, the author determines the field generated within a guide of arbitrary section by a prescribed tangential electric field in the wall. This field is expressible in terms of two "generating functions" [axial components of the electric and magnetic vector potentials], each of which is developed into a series of corresponding eigenfunctions, the coefficients involving integrals containing the tangential electric field at the wall. Referring to the analogy of the slot problem with that of scattering by a narrow metal strip [Babinet's principle; cf. H. G. Booker, J. Inst. Elec. Engrs. Part IIIA, 93, 620–626 (1946); E. T. Copson, Proc. Roy. Soc. London. Ser. A. 186, 100–118 (1946); these Rev. 8, 179], the author points out that only the longitudinal component of **H** and the transverse component of **E** in the slot are of importance. Further, the "voltage" distribution along the slot is approximately sinusoidal near resonance. Evidently an incident wave in the guide will be scattered backwards and forwards by the slot, while some of the energy escapes through the slot into free space. The reflection and transmission coefficients are easy to express in terms of the voltage amplitude of the slot and the longitudinal component of **H** in the slot of the undisturbed H_{01}-wave. However, by far the most complicated problem is to determine the voltage amplitude at the slot. Assuming that the slot radiates into free space as though the relevant side of the wave guide were infinite in any direction, the author derives an integral equation from the condition that the longitudinal magnetic component shall be continuous in passing the slot from inside to outside the guide. The required voltage amplitude is obtained upon approximately solving the integral equation in a manner similar to that of Hallén [Nova Acta Soc. Sci. Upsaliensis 11, no. 4 (1938)] applied to antenna problems. The results of theory are expressed in terms of equivalent circuit elements. The greater part of the author's results have already been announced elsewhere [W. H. Watson, The Physical Principles of Wave Guide Transmission and Antenna Systems, Oxford, 1947; these Rev. 9, 125]. *C. J. Bouwkamp* (Eindhoven).

Stevenson, A. F. Relations between the transmitting and receiving properties of antennas. Quart. Appl. Math. **5**, 369–384 (1948).

Relations existing between transmitting and receiving properties of antennas have been known for some time [J. R. Carson, Bell System Tech. J. **3**, 393–399 (1924); **9**, 325–331 (1930); Proc. Inst. Radio Engrs. **17**, 952–956 (1929); A. Sommerfeld, Z. Hochfrequenztech. **26**, 93–98 (1925); S. Ballantine, Proc. Inst. Radio Engrs. **17**, 929–951 (1929)]. Previous discussions of the matter are open to various objections, which are listed by the author. He aims at a discussion that is more thorough and rigorous than any thus far given. His analysis leads to the formulation of three fundamental mathematical problems, associated with a transmitting, a parasitic and a receiving antenna, respectively. The solution of the last problem is simply expressible in terms of those of the other two [Thévenin's theorem]. A connection is established between the transmission and parasite problems under the restrictions: (1) the incident field can be replaced by the field of an equivalent Hertzian doublet, (2) the medium in the neighborhood of this doublet is homogeneous and isotropic. So far, the discussion by the author is novel, in that relations are established between the performance of a single antenna when acting as transmitter, parasite or receiver. The more usual results of Carson and Sommerfeld relate to two antennas. The precise formulation of this reciprocity theorem is also given. Other subjects treated are (1) the influence of the geometry of the antenna gap, (2) whether or not the directional properties of an antenna are identical for transmission and reception, (3) the influence of imperfect conductivity of the antenna, (4) the extension of Lorentz's reciprocity theorem to anisotropic media. *C. J. Bouwkamp* (Eindhoven).

Tai, C. T. Coupled antennas. Proc. I. R. E. **36**, 487–500 (1948).

The author applies the King-Middleton modification of Hallén's solution of the antenna problem [Quart. Appl. Math. **3**, 302–335 (1946); these Rev. **7**, 401] to the case of two coupled antennas of identical size. This problem was originally treated by King and Harrison [J. Appl. Phys. **15**, 481–495 (1944); these Rev. **6**, 222] using the Hallén parameter; the present solution improves further on theirs by including the term corresponding to the contribution to the vector potential from the second antenna as part of the main integral, rather than as a correction term. For two closely-coupled antennas driven anti-symmetrically it is found that the solution is equivalent to that obtained from conventional transmission line theory. *M. C. Gray.*

Smith, P. D. P. The conical dipole of wide angle. J. Appl. Phys. **19**, 11–23 (1948).

The author gives an alternative treatment of the Schelkunoff formulation of the biconical antenna problem [Proc. I. R. E. **34**, 23P–32P (1946); these Rev. **7**, 402]. The input admittance for any cone angle can be expressed as the sum of an infinite series of Legendre functions whose coefficients are defined by an infinite set of linear equations. Use of the orthogonal properties of solutions of Legendre's equation makes it possible to obtain numerical results without actually determining the appropriate Legendre functions. As an approximate solution the author evaluates the coefficients when the principal wave and the first complementary wave in the antenna region are retained, together with the first two outside waves. *M. C. Gray* (New York, N. Y.).

Kuznecov, P. I. The propagation of electromagnetic waves along a line. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 615–620 (1947). (Russian)

The problem of the propagation of electromagnetic waves along a linear conductor is stated in the form of the solution of a system of two linear partial differential equations for the potential $V(x, t)$ and the current $I(x, t)$ together with the boundary conditions $V(x, 0)=0$, $I(x, 0)=0$. The solution is obtained by means of the Laplace transformation and its inversion. The results are expressed in terms of integrals which are known to be expressible in terms of Lommel's and Bessel's functions. A quite complete list of references, dealing with the same problem, is given. It is stated that in these earlier papers the solution of this problem was given in terms of definite integrals which had to be evaluated by approximation methods. *H. P. Thielman.*

Graffi, Dario. Sulle equazioni per la propagazione delle perturbazioni elettromagnetiche lungo fili conduttori. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **1**, 105–111 (1944).

Pollaczek, F. Théorie du câble coaxial. I. J. Phys. Radium (8) **8**, 215–224 (1947).

Pollaczek, F. Théorie du câble coaxial. II. J. Phys. Radium (8) **8**, 244–251 (1947).

The author develops a theory of the coaxial cable based on the Maxwell equations, rather than on transmission line theory. Using cylindrical coordinates, he assumes a wave which is transverse magnetic, sinusoidal in time, and with circular symmetry. The only nonzero components of the electric and magnetic vectors are E_z, E_r, and H_ϕ, and each of these is a product of $\exp(i\omega t-\gamma z)$ by a function of r expressible in terms of Bessel functions of orders 0 and 1. Solutions are sought for four regions: within the conductors, between them and exterior to the outer conductor. The complex propagation constant γ plays the role of an eigenvalue, with both a discrete and continuous spectrum for some of the regions. Each region has its own values of the constants ϵ, μ, σ (dielectric constant, permeability, conductivity). The mathematical methods used include contour integration, Green's functions, integral equations and expansions in series of orthogonal functions. The author considers some special cases of the field produced by a given source for a cable of finite length and for a semi-infinite cable. *O. Frink* (State College, Pa.).

Tucker, D. G. A note on Duhamel's integral, with particular reference to the transient response of cascaded filter sections. Philos. Mag. (7) **39**, 203–212 (1948).

The author demonstrates that the solution of transient response problems in cascaded circuits may be facilitated by the use of power series to represent the response of the individual circuits. This method is suggested for those problems in which the Duhamel integral offers technical difficulties. *A. E. Heins* (Pittsburgh, Pa.).

Stumpers, F. L. H. M. On the calculation of impulse-noise transients in frequency-modulation receivers. Philips Research Rep. **2**, 468–474 (1947).

The effect of an impulse-noise transient having the form of a rectangular pulse (which the author replaces by a Dirac δ-function) on a frequency-modulation receiver is calculated by means of a series expansion of the phase. For simplicity the intermediate frequency filter is assumed to be a single resonant circuit, but the method can be used with any

filter. If the amplitude $A(t)$ due to the transient of the input voltage to the frequency detector is smaller than the signal amplitude, then the phase is expanded in a series in terms of the powers $[A(t)]^n$ and then the effect of each term in the low-pass audio filter is calculated by means of Laplace transforms. If the impulse amplitude exceeds the signal amplitude, the series expansion is made in terms of the negative powers $[A(t)]^{-n}$. The formulas obtained explain the existence of the so-called impulse-noise threshold, and allow one to compute the gain for frequency modulation as compared to amplitude modulation for this type of noise.

O. Frink (State College, Pa.).

Wise, W. Howard. Potential coefficients for ground return circuits. Bell System Tech. J. **27**, 365–371 (2 plates) (1948).

The author derives an integral for the effect of the finite conductivity and dielectric constant of the earth on the potential coefficient of a one wire ground-return circuit, on the assumption that the current on the line is propagated without attenuation and at the velocity of light. The integral is approximated in terms of logarithmic integrals. Two accompanying graphs show numerical values of the integral for typical values of the physical constants. They were checked by a few direct numerical integrations.

C. J. Bouwkamp (Eindhoven).

Tellegen, B. D. H. The determination of the integration constants when calculating transient phenomena. Philips Research Rep. **3**, 24–36 (1948).

In linear bilateral networks, transient solutions for applied voltages which are discontinuous at time $t=0$ do not satisfy the differential equation at $t=0$. By repeated integrations, the author establishes relations expressing the discontinuities in the current and its derivatives in terms of those in the applied voltage and its derivatives. This leads to a new method of evaluation of integration constants; applications are given to the steady state solution of two simple circuits with discontinuous periodic voltages and currents applied to them.

E. Weber (Brooklyn, N. Y.).

Quantum Mechanics

***Flügge, Siegfried, und Marschall, Hans. Rechenmethoden der Quantentheorie dargestellt in Aufgaben und Lösungen. I. Teil: Elementare Quantenmechanik.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LIII. Springer-Verlag, Berlin and Göttingen, 1947. x+240 pp.

This book consists of a brief introduction, in which just enough of the principles of elementary quantum mechanics are stated to get the basic equations set up; seventy-nine sections, averaging about 2½ pages in length and each consisting of the statement of a problem and its solution; and a short mathematical supplement which contains condensed information on the simplest systems of curvilinear coordinates, a brief discussion of the solution of linear ordinary differential equations by series and a number of important formulas from the theory of the hypergeometric and confluent hypergeometric functions. The problems dealt with are mostly those whose solution involves the use of analysis, and mathematical results are rather habitually stated without proof. There are not more than half a dozen references to books or papers.

Topics in which a sufficient number of problems are solved include stationary states, illustrations of the meaning of certain operators, the traditional single-particle calculations using Dirac's equation and the case of two identical particles. The general many-body theory is omitted on the grounds that its discussion would have to depend heavily on mathematics other than analysis. Some of the methods of collision theory are fairly adequately illustrated, but Dirac's method of variation of parameters is by no means satisfactorily treated, and there is no mention of the important method of momentum space. No problem is solved involving perturbation theory applied to a degenerate system. Phase integral methods are not mentioned.

There are a number of mistakes and misleading statements, some of them ascribable to the effort to be brief, some of more serious nature. Two rather serious shortcomings which happened to strike the reviewer's attention may be mentioned. (a) The method of variation of parameters is not at all well explained, an average of the quantities $|a_n|^2$ being always referred to where the sum is required, and the meaning of the factor $(\Delta E_n)^{-1}$ being altogether erroneously stated [p. 132]. It would certainly not be possible for a student not already familiar with the method ever to make out for himself the relation between the working of the example in the next section and the preceding explanation of the method. (b) The nature of the invariance of the Dirac equation is completely misstated; the authors fall into the old trap of tacitly taking the wave function to be scalar, and this in spite of the fact that in the preceding section a two-dimensional example of the sort of invariance actually involved was worked out altogether correctly, albeit in needlessly pedestrian fashion.

There is a need for an inclusive and reliable collection of illustrative problems in this subject. The present attempt at one has considerable interest, and will be useful; but because of its spotty coverage and liability to nontrivial mistakes it should only be used by, or under the supervision of, an experienced teacher of the subject.

W. H. Furry.

Bargmann, V., and Wigner, E. P. Group theoretical discussion of relativistic wave equations. Proc. Nat. Acad. Sci. U. S. A. **34**, 211–223 (1948).

It has been shown [Wigner, Ann. of Math. (2) **40**, 149–204 (1939)] that a classification of all unitary representations of the inhomogeneous Lorentz group yields a classification of all possible relativistic wave equations for free particles. Hence, without any hypothesis about the form of such equations, it is possible to give an exhaustive list of the irreducible representations to which they correspond, and to show that every irreducible equation is equivalent to a set of differential equations. In this paper, for each such representation, there is given explicitly a differential equation the solutions of which transform according to that representation. These equations are then characterized by invariants constructed from the infinitesimal operators which generate these transformations.

H. C. Corben.

Ginsburg, V. L., and Tamm, I. E. On the theory of spin. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 227–237 (1947). (Russian)

In a relativistic theory, the state of a particle of spin s is ordinarily given by $2(2s+1)$ wave functions, and the particle, therefore, may be said to possess "internal" degrees of

freedom which are characterized by discrete variables. The authors investigate theories in which the internal degrees of freedom are described by continuous variables.

(I) As a first example, they treat the case where these variables are the components u_i of a space-like four-vector of constant length r. Let x_i $(1\leqq i\leqq 4)$ denote the space-time variables $(x_4=ict)$, set $\partial_i=\partial/\partial x_i$, $M_{ik}=u_k\partial/\partial u_i-u_i\partial/\partial u_k$, and $L_1=\frac{1}{2}M_{ik}M_{ik}$. The following equation is suggested:

$$(1)\qquad (\Box-\kappa^2+\beta L_1)\psi(x_i,u_i)=0,$$

where $\Box=\sum_i\partial_i^2$, β and κ are numerical constants and ψ is assumed to be a scalar function. Product solutions of the form $\psi(x_i,u_i)=\psi(x_i)\varphi(u_i)$ lead to the equations

$$(2)\qquad (\Box-\kappa^2+\lambda\beta)\psi(x_i)=0;\quad L_1\varphi=\lambda\varphi.$$

The parameter λ determines the rest mass of the particle, since $m_0^2c^2=\hbar^2\kappa^2-\lambda\beta$. On the hyperboloid $\sum_i u_i^2=r^2$, polar coordinates χ, θ, ϕ $(-\infty<\chi<\infty$, $0\leqq\theta\leqq\pi$, $0\leqq\phi\leqq 2\pi)$ are introduced: $u_1=r\cosh\chi\sin\theta\cos\phi$, $u_2=r\cosh\chi\sin\theta\sin\phi$, $u_3=r\cosh\chi\cos\theta$, $u_4=ir\sinh\chi$. Then

$$dS=\cosh^3\chi\cdot\sin\theta\,d\chi\,d\theta\,d\phi$$

is a Lorentz invariant volume element. To study the second equation (2) the authors determine the square integrable solutions (with respect to dS) of the form $\varphi(\chi,\theta,\phi)=R(\chi)\cdot Y_{lm}(\theta,\phi)$ (where Y_{lm} is a spherical harmonic). For each $l\geqq 1$, a sequence of functions R_{jl} is obtained such that (3a) $\lambda=1-j^2$, $j=1,2,\cdots$; (3b) $l=j,j+1,\cdots$, and the functions $R_{jl}Y_{lm}$ are orthogonal and normalized. If a square integrable function $\psi(\theta,\phi,\chi)$ may be expanded in a series (4) $\psi=\sum_{lmj}a_{lmj}Y_{lm}R_{jl}$, then $\int|\psi|^2dS=\sum_{lmj}|a_{lmj}|^2$. The Lorentz transformed function is then of the same form, and its square integral remains invariant, i.e., these functions transform according to a unitary infinite dimensional representation of the Lorentz group. [Reviewer's note. The authors neglect the positive continuous spectrum of the operator L_1 and make the erroneous assertion that every square integrable function orthogonal to Y_{00} is of the form (4). This oversight, however, hardly affects the main arguments of the discussion.] Since the equation (1) admits infinitely many values of the spin the authors postulate the additional equation (5) $(M_{ik}M_{ik}\partial_i\partial_k-(j_0+1)\Box)\psi=0$, j_0 being a constant. In the particle's rest system the equation (6) $l^2+l-j^2-j_0=0$ is shown to hold for a fixed value of the angular momentum. The possible spin values are those integers l which satisfy (6) and the condition $l\geqq j$ [cf. (3)]. Consequently, j_0 must be an integer. For a given j_0, the solution $l=j=j_0$ always exists. If $j_0\geqq 5$, a finite number of additional solutions may be found.

(II) The foregoing discussion may be considerably generalized by using the results of I. M. Gelfand and M. A. Neumark on the unitary (infinite dimensional) representations of the Lorentz group. [These results have meanwhile been published [Acad. Sci. USSR. J. Phys. **10**, 93–94 (1946); Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 411–504 (1947); these Rev. **8**, 132; **9**, 495]. Cf. also Harish-Chandra, Proc. Roy. Soc. London. Ser. A. **189**, 372–401 (1947); V. Bargmann, Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 132, 133.] The irreducible unitary representations are characterized by the equations

$$(7)\qquad \begin{cases} L_1\varphi\equiv\frac{1}{2}M_{ik}M_{ik}\varphi=\lambda_1\varphi, \\ L_2\varphi\equiv -i(M_{23}M_{14}+M_{31}M_{24}+M_{12}M_{34})\varphi=\lambda_2\varphi; \end{cases}$$

$$(7a)\qquad \lambda_1=1-j^2+\alpha^2,\quad \lambda_2=-\alpha j,$$

which hold for every vector φ in the representation space. The operators M_{ik} are the corresponding infinitesimal generators of the Lorentz group; their precise form depends on the particular realization which has been chosen, e.g., in terms of vector or spinor quantities. In (7a), α may be any real number, j is integral or half integral (including $j=0$). The equation (3b) remains valid; l characterizes the irreducible representations of the rotation group which occur. Two different sets of j, α lead to inequivalent representations. For the case discussed above, $\alpha=0$. [Reviewer's note. There exist also representations for $\lambda_2=0$, $0<\lambda_1<1$, which cannot be expressed in the form (7a).] If the equations (1) and (5) are retained, one obtains in the rest system $l^2+l-j^2-\delta=0$; $\delta=j_0-\alpha^2$. Here δ is positive and integral or half integral according as l (or j) is integral or half integral. Hence only a finite number of α- and j-values, and consequently only a finite number of spin values appear.

After briefly discussing possible variants of the equations (1) and (5) the authors mention a serious difficulty of the theory proposed. As shown by Fierz and Pauli [Proc. Roy. Soc. London. Ser. A. **173**, 211–232 (1939); these Rev. **1**, 190] the interaction with the electromagnetic field cannot be described by simply replacing $\partial/\partial x_k$ by the operator $\Pi_k=\partial/\partial x_k-(ie/\hbar c)A_k$ (A_k the components of the four-vector potential) if the spin of the particle is greater than one. Additional wave functions must be introduced, which vanish in the absence of the electromagnetic field. A similar difficulty arises here, and the authors have not yet succeeded in treating the interaction with the electromagnetic field in a satisfactory way. *V. Bargmann* (Princeton, N. J.).

Badarau, G., et Stihi, E. Sur la théorie quantique de la désintegration radioactive. Acad. Roum. Bull. Sect. Sci. **24**, 297–300 (1943).

The author uses the theory of Gamow to calculate the constant of radioactive disintegration of a nucleus. He starts with a Schrödinger equation $H\Psi=\alpha\Psi$, where the potential energy term of H is constant within the nucleus and of Coulomb type without. Following Gamow, α is assumed to be complex, of the form $E-\lambda h/4\pi i$. Within the nucleus, Ψ is expanded in an infinite series of Bessel functions of half-integral order; outside the nucleus in an infinite series of Whittaker functions $W_{k,j+\frac{1}{2}}(br)$. The coefficients in the two series are determined by matching the series and their normal derivatives at $r=r_0$ (where r_0 is the nuclear radius). There results an exact though complicated formula involving Bessel functions, Whittaker functions and known constants. This formula relates the nuclear radius with the different possible values of E and λ. The physical interpretation of the formula is to be given in another paper.

O. Frink (State College, Pa.).

Schatzman, Evry. Valeurs propres et énergie d'interaction. Cahiers de Physique, no. **28**, 55–67 (1945).

The author considers the question: is it theoretically possible in quantum mechanics to determine exactly and uniquely the potential energy function for the interactions of a system of particles, from measurements of energy levels only? If the energy levels are nondegenerate, he shows this is possible. For degenerate energy levels, a knowledge of intensities, that is, of the spectral distribution of energy among the different states, is required also to determine the potential energy uniquely. The author's method is to treat the difference between the true and supposed potentials as a perturbation of the supposed (approximate) potential and to expand this perturbation in an infinite series of eigenfunctions of the unperturbed state. This is illustrated by

examples. It is pointed out that the energy levels and intensities needed to determine the interaction potential are simultaneously measurable. *O. Frink.*

Jost, Res. Über die falschen Nullstellen der Eigenwerte der S-Matrix. Helvetica Phys. Acta 20, 256–266 (1947).

This paper deals with the method of computing the stationary energy values (bound states) of a quantum mechanical system from its S-matrix [cf. W. Heisenberg, Z. Physik 120, 513–538, 673–702 (1943); C. Møller, Danske Vid. Selsk. Mat.-Fys. Medd. 23, no. 1 (1945); 22, no. 19 (1946); these Rev. 4, 292; 9, 70]. The author discusses the simple example of two interacting particles according to the non-relativistic theory. If the motion of the center of gravity is separated, the relative motion is described by the wave equation (1) $d^2\phi/dr^2+k^2\phi=V(r)\phi$, assuming a central field of force and the angular momentum zero. Here $\psi=r\cdot\phi$ and the units are so chosen that the energy equals k^2. The admissible solutions of (1) vanish at $r=0$. In case k is real, ϕ has, for large r, the asymptotic form (2) $\phi\sim C(e^{ikr}-S(k)e^{-ikr})$ provided that $V(r)$ vanishes rapidly enough as $r\to\infty$ [cf. (6) below]. Here $S(k)$ has the absolute value 1, and is a proper value of the S-matrix. If $S(k)$ is an analytic function it may be continued to imaginary values of k. Assuming the imaginary part $\Im(k)$ of k to be negative, $S(k_0)$ must vanish for a bound state of energy k_0^2. Then, according to H. A. Kramers, (3) $-idS(k)/dk|_{k=k_0}>0$ [cf. Møller's second paper]. [By discussing a potential of the form $V(r)=a\cdot e^{-br}$, S. T. Ma, Physical Rev. (2) 69, 668 (1946), has previously shown that a zero of the function $S(k)$ does not necessarily correspond to a bound state.]

The author investigates the occurrence of "false" or "redundant" zeros in a systematic way. Let $f(k, r)$ (k real) be that solution of (1) which has the asymptotic form (4) $f(k, r)\sim e^{-ikr}$ for large r, and set $f(k, 0)=f(k)$. The solution which vanishes at $r=0$ is then given by

(5) $\phi(k, r)=f(k, r)-S(k)f(-k, r)$; $S(k)=f(k)/f(-k)$.

Assuming that, with suitable positive constants M, ϵ, δ, (6) $|V(r)|<M\cdot(r+\epsilon)^{-(2+\delta)}$ (this condition may be considerably relaxed) the author establishes the following statements. (a) For $\Im(k)\leqq 0$, $f(k, r)$ is uniquely determined by (4), and it is a regular analytic function in k for $\Im(k)<0$; for $\Im(k)>0$ it is defined by analytic continuation. (b) If, for any fixed r_0, $f(k, r_0)$ and $f'(k, r_0)$ (differentiation with respect to r) are regular in k at some value k_0, then $f(k, r)$ and $f'(k, r)$ are regular at k_0 for all r. (c) A necessary and sufficient condition for k_0 to be a "true" zero of $S(k)$ (to which a bound state corresponds) is the vanishing of $f(k_0)$. If $f(k_0)\neq 0$, and $f(-k_0)=\infty$, k_0 is a "false" zero, and $\phi(k_0, r)$ [cf. (5)] vanishes identically in r. (d) It follows from (b) that no false zeros appear if $V(r)=0$ for sufficiently large r, say, $r>r_0$, because then $f(k, r)=e^{-ikr}$ for $r>r_0$, and hence is regular in the whole complex k-plane. This explains the success of "cut off" procedures.

By discussing specific examples the author shows, in particular, that the condition (3) does not suffice to rule out false zeros, so that as yet there exists no method for computing the stationary energy values from the S-matrix, i.e., from the function $S(k)$. An interesting example is provided by the potential $V(r)=\alpha_1 e^{-\lambda_1 r}+\alpha_2 e^{-\lambda_2 r}$ ($\alpha_i\neq 0$, λ_1/λ_2 irrational). False zeros appear at the points

$$k=-\tfrac{1}{2}i(\mu_1\lambda_1+\mu_2\lambda_2)\quad(\mu_1, \mu_2=0, 1, 2, \cdots;\ \mu_1^2+\mu_2^2>0).$$

V. Bargmann (Princeton, N. J.).

Destouches-Février, Paulette. Les forces nucléaires et le principe des trièdres respectifs. C. R. Acad. Sci. Paris 226, 878–880 (1948).

Destouches-Février, Paulette. Relations d'incertitude liées à la complémentarité corpuscules-système de Louis de Broglie. C. R. Acad. Sci. Paris 226, 468–470 (1948).

In a system with strong interactions, the total mass M of the system and the mass m of a single constituent are complementary quantities in the sense of de Broglie. The corresponding operators do not commute. The author derives the uncertainty relation $\sigma_m\sigma_M\geqq k(h/c\tau_0)^2$, where k is a numerical factor, τ_0 is Heisenberg's "minimal length" and σ_m and σ_M are the standard deviations of simultaneous measurements of m and M. *O. Frink.*

Destouches, Jean-Louis, et Cazin, Michel. Corrélation entre la complémentarité de Louis de Broglie et la matrice caractéristique de Heisenberg. C. R. Acad. Sci. Paris 226, 566–567 (1948).

This compares briefly certain aspects of de Broglie's notion of the complementarity of particles and systems of particles [see the preceding review] with Heisenberg's concept of characteristic matrix [Z. Physik 120, 513–538, 673–702 (1943); these Rev. 4, 292]. *O. Frink.*

Destouches-Février, Paulette. Repères fondamentaux et notion de masse en mécanique classique et en mécanique ondulatoire. C. R. Acad. Sci. Paris 226, 635–637 (1948).

For a universe consisting of a finite number of mass particles in wave mechanics, the author distinguishes three natural frames of reference: a real frame with origin at the center of gravity G of the universe and two apparent frames with origins at $\bar{G}$, the mean position of G. Measurements by observers are macroscopic and should be referred to one of the mean frames of reference. Hence it is to be expected that the observed laws of interaction between two particles (or between a particle and a photon) should contain as a factor a Gauss error function like $\exp(-kr^2/\sigma^2)$, where σ is the standard deviation of G relative to $\bar{G}$. *O. Frink.*

Destouches, Jean-Louis. Répercussion du principe des trièdres respectifs sur les lois d'interaction entre la matière et le rayonnement. C. R. Acad. Sci. Paris 226, 639–641 (1948).

Using the principle of mean frames of reference [see the preceding review] the author shows that the interaction term in the Hamiltonian of the electron-photon system becomes modified by the factor $(\pi/2)^{\frac{1}{2}}\sigma^3 q\exp(-k^2\sigma^2/8)$, where k is the magnitude of the momentum of the photon, and σ is the standard deviation of the center of gravity of the universe (or σ may be identified with Heisenberg's minimal length or Eddington's uncertainty constant). He also derives in terms of σ an explicit expression (as a Fourier sine transform) for the electrostatic interaction potential of two charged particles. In both formulas the divergences and infinite energies previously encountered are avoided, since a Dirac δ-function is replaced by a Gauss error function involving σ. *O. Frink* (State College, Pa.).

Cazin, Michel. Interaction entre matière et rayonnement, constante d'incertitude et énergie propre de l'électron. C. R. Acad. Sci. Paris 226, 641–643 (1948).

The author separates into two terms the interaction potential of two electrons obtained from the theory of J. L. Destouches [see the preceding review]. An exact formula is

given for the first term, which corresponds to a potential of coincidence. An approximate formula is derived for the other term, which corresponds to a Coulomb repulsion. From this the author obtains, on the basis of some natural assumptions, the approximate formula $(2/\pi)^{\frac{1}{2}}e^2/\sigma$ for the self-energy of an electron. Identifying this self-energy with mc^2 gives the value 11×10^{-14} for the quantity σ occurring in the theory of Destouches, which may be identified with Eddington's constant of uncertainty. Eddington's value of this constant was 9.6×10^{-14}, computed from his supposed values for the radius of the universe and the number of particles in the universe. The agreement is good in view of the uncertainty of Eddington's cosmological constants.

O. Frink (State College, Pa.).

Tortrat, A. Sur les relations d'incertitude de Heisenberg. C. R. Acad. Sci. Paris **225**, 1280–1282 (1947).

Using the notion of quadric of dispersion in Hilbert space, the author derives for the nth energy level of the harmonic oscillator the uncertainty relation $\Delta p\Delta q=(n+\frac{1}{2})h/2\pi$ as an improvement on the approximate Heisenberg relation $\Delta p\Delta q\sim nh/2\pi$. *O. Frink* (State College, Pa.).

Faure, Robert. Correspondance mécanique classique, mécanique ondulatoire; intégrale du deuxième ordre indépendante du temps. C. R. Acad. Sci. Paris **225**, 1279–1280 (1947).

Faure, Robert. Correspondance mécanique classique-mécanique ondulatoire. Intégrale du deuxième ordre indépendante du temps: conditions d'existence. C. R. Acad. Sci. Paris **226**, 1174–1175 (1948).

Faure, Robert. Correspondance mécanique classique-mécanique ondulatoire. Intégrale du deuxième ordre indépendante du temps. Étude de deux cas particuliers. C. R. Acad. Sci. Paris **226**, 1506–1508 (1948).

If a classical mechanical system with Hamiltonian $H=\frac{1}{2}\sum A_ip_i^2+u$ has a first integral of the form $F=\sum P_ip_i^2+R$, the related quantum mechanical system may or may not have a corresponding first integral. The author finds conditions involving the quantities A_i and P_i which are necessary and sufficient for the existence of this quantum mechanical first integral. The results are applied to two cases of mechanical systems with two and three degrees of freedom.

O. Frink (State College, Pa.).

Viard, Jeannine. Le calcul vectoriel gauche et l'invariance relativiste de l'équation de Dirac. C. R. Acad. Sci. Paris **226**, 637–639 (1948).

In the noncommutative vector calculus previously discussed by the author [M. Cazin and J. Viard, same C. R. **224**, 452–454 (1947); these Rev. **8**, 553] the Dirac equation is changed by a Lorentz transformation from a commutative to a noncommutative frame of reference; that is, it is no longer relativistically invariant. Two cases where such transformations are naturally encountered are discussed.

O. Frink (State College, Pa.).

Slansky, Serge. La notion de centre de gravité en mécanique ondulatoire relativiste. J. Phys. Radium (8) **8**, 56–60 (1947).

In nonrelativistic quantum theory the following two assertions about a system of N free particles are known to be true. (1) The Hamiltonian may be decomposed into two parts, the first being the kinetic energy of the center of gravity, and the second depending only on the relative motion of the particles. (2) The Schrödinger equation admits a solution of the form $\psi(X, Y, Z, t)$, where X, Y, Z are the coordinates of the center of gravity. In relativistic quantum theory, (1) does not hold any longer, but (2) may be taken over. In order to define the center of gravity in a Lorentz invariant fashion, the author ascribes to each particle a separate time coordinate, and sets

$$X=\sum\mu_ix_i/\sum\mu_i,\quad Y=\sum\mu_iy_i/\sum\mu_i,$$
$$Z=\sum\mu_iz_i/\sum\mu_i,\quad T=\sum\mu_it_i/\sum\mu_i$$

$(1\leqq i\leqq N)$, where (x_i, y_i, z_i, t_i) are the coordinates of the ith particle and the μ_i are N constants. Let m_i be the mass of the ith particle. The author first shows that (2) holds for N free particles of spin $\frac{1}{2}$ if the absolute values of all the ratios μ_i/m_i are equal to each other, in particular if $\mu_i=m_i$. This result remains valid for particles of higher spin, because the corresponding wave equations may be obtained from those for particles of spin $\frac{1}{2}$. *V. Bargmann.*

Arnous, Edmond. Quelques applications de la théorie des groupes de transformations unitaires, en calcul des probabilités et en mécanique quantique. Revue Sci. **85**, 553–558 (1947).

The author presents a qualitative discussion (without theorems or proofs) of the role of one-parameter groups of unitary operators in the theories of stochastic processes and quantum mechanics. The purpose of the paper is, apparently, to expound some basic (and mathematically well-known) concepts to physicists. The author considers also the situation in which the parameter space is a locally compact Abelian group and mentions some analogies between the theory of characters and such physical concepts as quantization and the principle of indeterminacy.

P. R. Halmos (Chicago, Ill.).

Ţiţeica, Ş. Sur le temps propre en mécanique ondulatoire. Acad. Roum. Bull. Sect. Sci. **25**, 189–192 (1943).

***Wentzel, Gregor. Einführung in die Quantentheorie der Wellenfelder.** Franz Deuticke, Wien, 1943; J. W. Edwards, Ann Arbor, Mich., 1946. iv+209 pp. $5.50.

This is the first book to be published dealing primarily with the fundamental principles on which theories of quantized fields are constructed. In it one finds careful consideration of the conservation laws, the various types of gauge transformations and proofs of Lorentz invariance. There is thorough discussion of cases in which certain coordinates and momenta are not actually independent and of ways of dealing with this situation. The order in which the various specific theories are discussed is not the historical one, but is better from the pedagogical point of view: first the scalar meson, then the vector meson, followed by quantum electrodynamics and finally the Dirac electron theory. The concluding chapter contains a descriptive account of some theories of particles of higher spin, a discussion of the relation between spin and statistics and some general remarks on the possible future development of the theory.

Being mainly devoted to questions of fundamental theory, the book contains only a few examples of actual calculations of such things as nuclear forces and scattering cross-sections. In some of the examples the method of canonical transformation is emphasized rather than the perturbation method; this is a very good feature. From the point of view of the reader interested in applications to the theory of nuclear forces, an outstanding omission is any discussion, or even

mention, of the pseudoscalar meson. Actually this is a natural omission, since the field theory of this particle would by itself contain nothing new, and it is not the author's purpose to explore all types of interactions.

The whole book is very well written, remarkably free from errors and easy to study. In its subject matter and point of view, particularly the discussion of time-independent and time-dependent operators and of the many-time theory, it is ideally suited for an approach to the recent new formulations of electron theory. *W. H. Furry.*

d'Espagnat, Bernard. Les fonctions caractéristiques dans la théorie quantique des champs. C. R. Acad. Sci. Paris **226**, 1175–1177 (1948).

The author extends his method of treating the harmonic oscillator, using characteristic functions [same C. R. **226**, 316–318 (1948); these Rev. **9**, 349] to more general systems whose Hamiltonian is quadratic in the operators corresponding to both coordinates and momenta, and in particular to the quantum theory of fields. *O. Frink.*

Green, Alex E. S. Higher order field equations. Physical Rev. (2) **73**, 519 (1948).

The present note reports the properties of a field whose field equation is of the eighth order. A Lagrangian, quadratic in the derivatives of the field potentials and including derivatives up to the fourth order, is considered. The field equation is shown to be $[\prod_{s=1}^{4}(a^2\Box-s_s^2)]\varphi_n(x,t)=0$. The energy-momentum tensor is constructed and the field is subsequently quantized. The interaction energy is also calculated. It is pointed out that the self-energy term is finite in the fourth-, sixth- and eighth-order field theories [see also A. E. S. Green, Physical Rev. (2) **72**, 628–631 (1947); these Rev. **9**, 69]. *C. Kikuchi* (East Lansing, Mich.).

Pomeranchuk, I. Generalization of the λ-limiting process and nonuniqueness of the elimination of divergencies in the quantum theory of elementary particles. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 667–674 (1947). (Russian)

In the case of a quantized scalar field φ the λ-limiting process consists in replacing the commutation rules (for two world points $\hat{r}_1$, $\hat{r}_2$)

$$[\varphi(\hat{r}_1),\ \varphi(\hat{r}_2)]=\Delta_1(\hat{r}_1-\hat{r}_2)=(2\pi)^{-3}\int\frac{\sin(\hat{k},\hat{r}_1-\hat{r}_2)}{\omega(k)}d\mathbf{k}$$

by

$$(1)\quad\begin{cases}[\varphi(\hat{r}_1),\ \varphi(\hat{r}_2)]=\frac{1}{2}\{\Delta_1(\hat{r}_1-\hat{r}_2+\hat{\lambda})+\Delta_1(\hat{r}_1-\hat{r}_2-\hat{\lambda})\}\\ \qquad=(2\pi)^{-3}\int\cos(\hat{k}\hat{\lambda})\,\dfrac{\sin(\hat{k},\hat{r}_1-\hat{r}_2)}{\omega(k)}d\mathbf{k}.\end{cases}$$

[For this formulation cf. Dirac, Ann. Inst. H. Poincaré **9**, 13–49 (1939); Pauli, Rev. Modern Physics **15**, 175–207 (1943); these Rev. **1**, 94; **5**, 277.] Here $\hat{a}\hat{b}$ is the scalar product of two four-vectors $\hat{a},\hat{b}$, ; $\omega(k)=k^2+\mu^2$, where μ is the rest mass of the field quanta; $\hat{\lambda}$ is a time-like four-vector which, in the computation of the physically relevant quantities, is made to converge to zero. Thus integrals of the type $\int_0^\infty k^s dk$ are replaced by

$$(2)\qquad I_s=\lim_{\lambda_4\to 0}\int_0^\infty k^s\cos(k\lambda_4)dk.$$

As $I_{2n}=0$, but $I_{2n+1}=\infty$, the λ-limiting process removes the "classical," but not the specifically quantum mechanical divergencies. Dirac's attempt to remove these by introducing negative probabilities, and thereby replacing in (2) the limits of integration by $(-\infty,+\infty)$ [cf. Dirac, Proc. Roy. Soc. London. Ser. A. **180**, 1–40 (1942); these Rev. **5**, 277; W. Pauli, loc. cit.] gives rise to difficulties in the interpretation of the results of the theory and, moreover, does not lead to the elimination of the logarithmic divergencies which appear in the electron theory.

The author investigates the possibility of using a more general convergence factor without introducing negative probabilities. He replaces, in the integral in (1), $\cos(\hat{k}\hat{\lambda})$ by $D=\sum_n A_n(\hat{\lambda}_n)\cos(\hat{k}\hat{\lambda}_n)$, where $\hat{\lambda}_n$ are time-like four-vectors and A_n certain functions to be determined. Consequently, the commutation rules

$$[\varphi(\hat{r}_1),\ \varphi(\hat{r}_2)]=\tfrac{1}{2}\textstyle\sum_n A_n(\hat{\lambda}_n)\{\Delta_1(\hat{r}_1-\hat{r}_2+\hat{\lambda}_n)+\Delta_1(\hat{r}_1-\hat{r}_2-\hat{\lambda}_n)\}$$

are obtained. The main results of the discussion are the following. (a) The modified integrals I_{2n} remain zero. (b) By an actual construction the author shows that it is possible to choose the vectors $\hat{\lambda}_n$ and the functions A_n so as to annul the first p integrals I_{2m+1} $(0\leqq m\leqq p-1)$, and even to remove the logarithmic divergencies. (c) It is also possible, however, to obtain arbitrary nonvanishing values for the I_{2n+1}. This constitutes the essential "nonuniqueness" of the proposed procedure. (d) Additional terms depending on space-like vectors $\hat{\mu}_n$ may be added to D. The limits obtained for the integrals I_{2n+1} depend then on the particular way the $\hat{\mu}_n$ tend to zero, which, in the author's opinion, seems to indicate that space-like vectors should not be used. *V. Bargmann* (Princeton, N. J.).

Harish-Chandra. Relativistic equations for elementary particles. Proc. Roy. Soc. London. Ser. A. **192**, 195–218 (1948).

The wave equations of an elementary particle are assumed to be of the form $i\gamma^k\partial\psi/\partial x^k+\chi\psi=0$, where the γ^k are a set of four $n\times n$ irreducible matrices and χ is a constant. It is further assumed that (i) each component of ψ satisfies the second order wave equation $g^{kl}\partial^2\psi/\partial x^k\partial x^l+\chi^2\psi=0$ and (ii) either the total charge or the total energy of the field associated with the particle is positive definite. It is shown that as a consequence of (ii) the theory can be quantized in the case of no interaction with an external electromagnetic field. When an interaction is present it is an open question as to whether the quantization can be performed. The nonrelativistic approximation is carried out and it is shown that a particle exhibits a pure magnetic moment in this approximation. The paper concludes with the theory of a particle which has states of spin both $\frac{3}{2}$ and $\frac{1}{2}$. The equations of this particle are given in terms of tensors and two-component spinors. *A. H. Taub* (Princeton, N. J.).

Nilsson, S. Bertil. On the calculation of self-energies in quantum theory by analytic continuation. Physical Rev. (2) **73**, 903–909 (1948).

T. Gustafson's method [Ark. Mat. Astr. Fys. **34A**, no. 2 (1946); these Rev. **8**, 427] for the electron self-energy calculations is extended by taking retardation into account. It is shown that the total self-energy is zero for the one-electron theory but logarithmically divergent in the hole theory. *C. Kikuchi* (East Lansing, Mich.).

Serpe, J. La théorie convergente de Dirac et le moment magnétique du proton et du neutron. Bull. Soc. Roy. Sci. Liége **16**, 38–51 (1947).

Jauch [Physical Rev. (2) **63**, 334–342 (1943); these Rev. **4**, 235] has shown that Dirac's method of field quantization

[Proc. Roy. Soc. London. Ser. A. 180, 1–40 (1942); these Rev. 5, 277] applied to the pseudoscalar and vector meson theories leads to wrong signs for the anomalous magnetic moments of neutron and proton, unless the vector meson is assumed to carry an anomalous magnetic moment of about three meson magnetons. The present author takes the trouble to demonstrate that, irrespective of this assumption, the sum of the anomalous neutron and proton moments (experimentally -0.12 nuclear magnetons) comes out with the wrong sign if the Dirac method is applied to the symmetrical Møller-Rosenfeld theory. Thus, to account for the neutron and proton moments on these lines, one needs two new constants: besides the anomalous meson moment a quantity defining the deviation from a strictly symmetrical theory. *L. Hulthén* (Lund).

Tamm, I. E. On certain mathematical methods of the theory of the scattering of particles. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 337–345 (1948). (Russian)

A variation method is developed for the determination of the phase shifts and partial cross-sections in the problem of the scattering of particles from a spherically symmetrical potential field $V(r)$ which satisfies

$$V(r)=O(r^{-1}),\ r\to 0;\quad V(r)=o(r^{-2}),\ r\to\infty.$$

By writing the radial wave-function associated with a spherical harmonic of order l in the form

$$R_l(r)=Cr^{-\frac{1}{2}}\{J_{l+\frac{1}{2}}(kr)+u_l(r)J_{-l-\frac{1}{2}}(kr)\}$$

the author shows that the Schrödinger equation is equivalent to a variation problem involving the function $u_l(r)$. The limiting value of u_l for $r\to\infty$ determines the phase shift. The relation between this method and the variation method devised by Hulthén [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 21 (1944); these Rev. **6**, 111] is examined. The transformation of the radial wave equation for $l=0$ into momentum space leads to an integral equation which is singular for $k=k_0$. A procedure is developed by which the solution of the singular equation is reduced to the solution of an integral equation without singularities. *W. H. Furry.*

Potier, Robert. Sur les propriétés du champ mésonique. C. R. Acad. Sci. Paris **226**, 1690–1692 (1948).

Petiau, Gérard. Sur les relations tensorielles entre densités de valeurs moyennes en théorie de l'électron de Dirac. II. Relations différentielles. J. Math. Pures Appl. (9) **26**, 1–14 (1947).

[For part I see the same J. (9) **25**, 335–346 (1947); these Rev. **9**, 468.] The author derives a set of tensor identities for tensors built up from the four-component spinors ψ satisfying the Dirac equations. These tensors are bilinear expressions in the spinor ψ and the derivatives of the complex conjugates of ψ with respect to the coordinates, $\bar{\psi}_{,\mu}$. The coefficients of the bilinear forms are built up from the Dirac matrices γ_μ in the well-known manner. The tensor identities given in the paper are those involving pairs of tensors.
A. H. Taub (Princeton, N. J.).

Petiau, Gérard. Sur les équations de propagation des ondes de la théorie de l'électron de Dirac dans un champ magnétique. C. R. Acad. Sci. Paris **226**, 1687–1689 (1948).

Novac, Valeriu. Sur la théorie du photon. Acad. Roum. Bull. Sect. Sci. **23**, 351–357 (1942).

The formulation of de Broglie's theory of photons is in terms of binary four-component spinors is reviewed. A Lagrangian from which these equations may be derived is given. The second quantization of the field is outlined and applied to the problem of Compton scattering. The scattering cross section is shown to go over to the Klein-Nishina one as the rest mass of the photon is set equal to zero.
A. H. Taub (Princeton, N. J.).

Dirac, P. A. M. On the theory of point electrons. Philos. Mag. (7) **39**, 31–34 (1948).

The author answers some criticisms which T. Lewis [Philos. Mag. **36**, 533–541 (1945); these Rev. **7**, 539] has made of the author's work on the Hamiltonian formulation of the classical equations of motion for point electrons. Dirac answers Lewis's criticisms concerning the proper time variables by giving a detailed explanation of the role played by these variables. *A. H. Taub* (Princeton, N. J.).

Lewis, T. On the theory of point electrons. Philos. Mag. (7) **39**, 116–122 (1948).

The author reviews the criticisms he made earlier [same Mag. (7) **36**, 533–541 (1945); these Rev. **7**, 539] which were answered by Dirac [cf. the preceding review]. Objection is no longer taken to the mathematical formulation. However, a different physical interpretation is proposed.
A. H. Taub (Princeton, N. J.).

Sokolov, A. A. On the theory of the point electron. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 280–284 (1948). (Russian)

Kahan, Théo. La matrice caractéristique de Heisenberg et la théorie des particules élémentaires. Revue Sci. **85**, 997–1002 (1947).

An expository lecture.

Drăganu, Mircea. Sur l'énergie propre de l'électron et l'introduction d'une longueur fondamentale en mécanique quantique. Acad. Roum. Bull. Sect. Sci. **24**, 94–95 (1943).

Tonnelat, Marie-Antoinette. Remarques sur la fusion de deux particules de Dirac. C. R. Acad. Sci. Paris **226**, 783–785 (1948).

Radicati, L. A. La funzione di Lagrange di alcune particelle elementari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 351–356 (1947).

Gião, Antonio. Théorie des particules fondamentales. I. Particules élémentaires. Portugaliae Math. **6**, 67–114 (1947).

The author's theories [for example, Gaz. Mat. Lisboa **7**, no. 30, 4–5 (1946); these Rev. **8**, 555] allow particles of matter with a spectrum of masses and particles of electricity with a spectrum of charges: here he supposes "fusion" of elementary particles to form heavier particles during a phase of contraction of the universe, and discusses spin momenta, magnetic moments, and uncertainty relations for these "microelectrons." The flexibility of a spectrum of subelectronic masses is applied to the theory of beta-ray spectra. Photons and the electromagnetic field are discussed.
C. Strachan (Aberdeen).